

Noncommutative Riemann Integration and Novikov–Shubin Invariants for Open Manifolds

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Given a C*-algebra \mathcal{A} with a semicontinuous semifinite trace τ acting on the Hilbert space \mathcal{H} , we define the family $\mathcal{A}^{\mathcal{R}}$ of bounded Riemann measurable elements w.r.t. τ as a suitable closure, à la Dedekind, of \mathcal{A} , in analogy with one of

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\mathcal{H} which are affiliated to \mathcal{A}'' and can be approximated in measure by operators in $\mathcal{A}^{\mathcal{R}}$, in analogy with unbounded Riemann integration. Unbounded Riemann measurable operators form a τ -a.e. bimodule on $\mathcal{A}^{\mathcal{R}}$, denoted by $\overline{\mathcal{A}^{\mathcal{R}}}$, and such a bimodule contains the functional calculi of selfadjoint elements of $\mathcal{A}^{\mathcal{R}}$ under unbounded Riemann measurable functions. Besides, τ extends to a bimodule trace on $\overline{\mathcal{A}^{\mathcal{R}}}$. Type II₁ singular traces for C*-algebras can be defined on the bimodule of unbounded Riemann-measurable operators. Noncommutative Riemann integration and singular traces for C*-algebras are then used to define Novikov–Shubin numbers for amenable open manifolds, to show their invariance under quasi-isometries, and to prove that they are (noncommutative) asymptotic dimensions.

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1. INTRODUCTION

In this paper we extend the notion of Riemann integrability to the non-abelian setting, namely, given a C*-algebra \mathcal{A} of operators acting on a Hilbert space \mathcal{H} , together with a semicontinuous semifinite trace τ , we define both bounded and unbounded Riemann measurable operators on \mathcal{H} with respect to τ . Then we apply the preceding construction in order to define the Novikov–Shubin numbers for amenable open manifolds and show that they can be interpreted as asymptotic dimensions.

Bounded Riemann measurable operators associated with (\mathcal{A}, τ) form a C*-algebra $\mathcal{A}^{\mathcal{R}}$, and τ extends to a semicontinuous semifinite trace on it. The unbounded Riemann measurable operators form a τ a.e. *-bimodule

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$\overline{\mathcal{A}^{\mathcal{R}}}$ on $\mathcal{A}^{\mathcal{R}}$ and τ extends to a positive bimodule-trace on $\overline{\mathcal{A}^{\mathcal{R}}}$. On the one hand, this leads to the notion of Riemann algebra as a C^* -algebra with some monotonic completion properties, and to the construction of the enveloping Riemann algebra for a pair (\mathcal{A}, τ) , in analogy with what is done for the von Neumann or Borel enveloping algebras [18]. On the other hand, Riemann integration provides a spatial theory of integration for C^* -algebras, namely the trace τ extends to natural classes of bounded and closed unbounded operators on the given Hilbert space \mathcal{H} .

The τ -a.e. bimodule $\overline{\mathcal{A}^{\mathcal{R}}}$ of unbounded Riemann measurable operators is a natural environment for the definition of singular traces, in particular for the type II_1 singular traces considered in [10]. Indeed, while type I singular traces (or singular traces at ∞) may be defined on bounded operators (and are anyhow determined by their restriction to bounded operators, cf. [10]), type II_1 singular traces (or singular traces at 0) need unbounded operators to be defined, since they vanish on the bounded ones. Therefore, we can define type II_1 singular traces for C^* -algebras by replacing the bimodule of unbounded measurable operators on a von Neumann algebra with the bimodule of unbounded Riemann measurable operators on a Riemann algebra.

As an application, and in fact a motivation, for our results on noncommutative Riemann integration, we study Novikov–Shubin numbers for amenable open manifolds. In fact, the trace on almost-local operators on the manifold defined in [12] can now be extended to Riemann measurable spectral projections. Therefore the spectral density function of the p -Laplacian is well defined, and we may define the Novikov–Shubin numbers and show they are invariant under quasi-isometries. As we shall see below, however, the noncommutative Riemann integration plays its major role in the dimensional interpretation of the Novikov–Shubin numbers. Indeed such interpretation is related to the existence of singular traces (at zero) depending on the spectral asymptotics (at zero) of the p -Laplacian. Unbounded Riemann integrable operators and singular traces on them furnish the necessary environment to prove such results.

The first part of this paper concerns noncommutative Riemann integration. The classical abstract Riemann integration (cf. [14, 23, 26]) is based on a topological space X (usually Hausdorff locally compact) and a Radon measure μ . Riemann measurable functions may be defined in two equivalent ways, either as functions which are discontinuous in a μ -negligible set or as functions which are L^1 approximated, both from above and from below, by continuous functions. The latter characterization may be rephrased as follows: Riemann measurable functions are separating elements of a Riemann-cut (\mathcal{R} -cut), where an \mathcal{R} -cut is a pair (A^-, A^+) of bounded classes of continuous functions in $C_0(X)$ s.t.

- $f^- \leq f^+$ for any $f^\pm \in A^\pm$
- $\forall \varepsilon > 0 \exists f_\varepsilon^\pm \in A^\pm : \int (f^+ - f^-) d\mu < \varepsilon$.

Replacing $(C_0(X), \mu)$ with (\mathcal{A}, τ) , where \mathcal{A} acts on \mathcal{H} , we define the family $\mathcal{A}^\mathcal{R}$ of bounded Riemann measurable elements on \mathcal{H} w.r.t. τ as the linear span of the set of separating elements in \mathcal{A}'' for \mathcal{R} -cuts in \mathcal{A} . $\mathcal{A}^\mathcal{R}$ will also be called the \mathcal{R} -closure of \mathcal{A} . Clearly τ uniquely extends to a positive functional on $\mathcal{A}^\mathcal{R}$.

Then we prove that $\mathcal{A}^\mathcal{R}$ is a C^* -algebra, the extension of τ to $\mathcal{A}^\mathcal{R}$ is a semicontinuous semifinite trace on it, $\mathcal{A}^\mathcal{R}$ is closed under functional calculus w.r.t. Riemann measurable functions, and is \mathcal{R} -closed, namely it is stable under \mathcal{R} -closure. We call Riemann algebra an \mathcal{R} -closed C^* -algebra, and enveloping Riemann algebra of \mathcal{A} the \mathcal{R} -closure of \mathcal{A} in the universal representation.

The main technical problem concerning bounded Riemann measurable elements arises in the definition already. Indeed we asked the classes (A^-, A^+) of the Riemann cut to be bounded. While such requirement is unnecessary in the abelian case, due to the simple order structure there, it was apparently not known if this was the case in the non-abelian setting either. The answer to this question is negative in general, as it is shown in Subsection 2.1 by an explicit counterexample, therefore the two corresponding Riemann closures (with or without boundedness assumption) are different in general. Moreover the non-uniformly bounded Riemann closure (\mathcal{U} -closure) is a C^* -algebra only when it coincides with the \mathcal{R} -closure. For these reasons we choose the \mathcal{R} -closure in the definition of Riemann algebra.

Unbounded Riemann measurable elements w.r.t. τ are defined as closed operators affiliated to \mathcal{A}'' which may be approximated in measure by elements of $\mathcal{A}^\mathcal{R}$. This definition is inspired by the corresponding construction by Christensen for Borel algebras [3]. The family $\overline{\mathcal{A}^\mathcal{R}}$ of such elements is a τ -a.e. $*$ -bimodule on $\mathcal{A}^\mathcal{R}$, namely it is closed under the $*$ -bimodule operations and the $*$ -bimodule properties hold τ almost everywhere. Even if $\mathcal{A}^\mathcal{R}$ is not necessarily closed w.r.t. the product operation (see Remark 3.16), given two operators $S, T \in \overline{\mathcal{A}^\mathcal{R}}$, there is $\tilde{T} \in \overline{\mathcal{A}^\mathcal{R}}$, equal to T τ -a.e., s.t. $S\tilde{T} \in \overline{\mathcal{A}^\mathcal{R}}$. Moreover, τ extends to a positive bimodule trace on it, satisfying $\tau(T^*eTe) = \tau(eTeT^*)$, for a suitable conull projection e . Functional calculi of selfadjoint elements of $\mathcal{A}^\mathcal{R}$ under unbounded Riemann measurable functions belong to $\overline{\mathcal{A}^\mathcal{R}}$.

In the last part of this paper we extend the definition of the Novikov–Shubin numbers to the case of amenable open manifolds.

As it is known, a general understanding of the geometric meaning of the Novikov–Shubin invariants is still lacking. We believe that the definition of these numbers as global invariants of an open manifold, rather than as

homotopy invariants of a compact one, may shed some light on their meaning. Our interpretation of Novikov–Shubin numbers as asymptotic dimensions goes in this direction.

This interpretation is based on a fundamental observation of Alain Connes, who showed that the integration on a d -dimensional compact Riemannian manifold may be reconstructed via the formula $\int f = \tau(f |D|^{-d})$, where τ is a singular trace, and D is the Dirac operator. Therefore, the dimension of a spectral triple in noncommutative geometry corresponds to the power of $|D|^{-1}$ giving rise to a non-trivial singular trace. This is analogous to the situation of geometric measure theory, where the dimension determines which power of the radius of a ball gives rise to a non trivial volume on the space.

It was shown in [11] that, as in the classical case, the Weyl asymptotics furnishes a dimension for a spectral triple via the formula

$$d = \left(\liminf_{n \rightarrow \infty} \frac{\log \mu_n}{\log 1/n} \right)^{-1},$$

where μ_n is the sequence of eigenvalues of $|D|^{-1}$, namely, when d is finite non-zero, there exists a singular trace (not necessarily a logarithmic one) which is finite non-zero on $|D|^{-d}$.

This formula makes sense also for non-compact manifolds, if one replaces the eigenvalue sequence μ_n with the eigenvalue function $\mu(t)$ of $\Delta_p^{-1/2}$ (see Section 4), and recovers the dimension of the manifold. But in this case, the behaviour for $t \rightarrow 0$ may be considered too, giving rise to an asymptotic counterpart of the dimension.

Here we show that, under suitable assumptions, the asymptotic dimension associated with the Laplacian Δ_p on p -forms coincides with the p th Novikov–Shubin number α_p , and that it behaves as a dimension in non-commutative geometry, namely $\Delta_p^{-\alpha_p/2}$ gives rise to a non-trivial singular trace on the unbounded Riemann measurable operators, which is finite nonzero on $\Delta_p^{-\alpha_p/2}$.

The singular traceability of $\Delta_p^{-\alpha_p/2}$ has already been proved in [11] in the case of Γ -coverings. The main problem in extending such dimensional interpretation to the case of amenable open manifolds is the fact that a normal trace on a von Neumann algebra of Γ -invariant operators is replaced by a semicontinuous trace on the C^* -algebra \mathcal{A}_p of almost local operators on p -forms defined in [12]. Indeed, since Novikov–Shubin numbers are defined in terms of the spectral asymptotics near zero of the p -Laplacian, the needed singular traces should be looked for among type II_1 singular traces [10]. Such traces however are defined on bimodules of unbounded operators, since they vanish on bounded ones, and the notion of unbounded operator affiliated to a C^* -algebra is too restrictive for our

purposes. Unbounded Riemann measurable operators affiliated with \mathcal{A}_p and type II_1 singular traces on them furnish the environment for the traceability statements, hence for the dimensional interpretation.

However, since the semicontinuous trace on \mathcal{A}_p is not normal with respect to the given representation of \mathcal{A}_p on the space of L^2 -differential forms, some assumptions are needed, as the vanishing of the torsion dimension introduced by Farber [8]; cf. Remark 5.4(c).

2. BOUNDED RIEMANN INTEGRATION

Let (\mathcal{A}, τ) be a C^* -algebra with a semicontinuous semifinite trace, acting on a Hilbert space \mathcal{H} . A pair (A^-, A^+) of sets in \mathcal{A}_{sa} is called an \mathcal{R} -cut (w.r.t. τ) if A^-, A^+ , are uniformly bounded, *separated*, namely for any pair $a^\pm \in A^\pm$ we have $a^- \leq a^+$, and τ -contiguous, namely

$$\forall \varepsilon > 0 \exists a_\varepsilon^\pm \in A^\pm, \quad \tau(a_\varepsilon^+ - a_\varepsilon^-) < \varepsilon. \quad (2.1)$$

An element $x \in \mathcal{A}''$ is called separating for the \mathcal{R} -cut (A^-, A^+) if it is selfadjoint and for any $a^\pm \in A^\pm$ we have $a^- \leq x \leq a^+$.

DEFINITION 2.1. Let (\mathcal{A}, τ) be a C^* -algebra with a semicontinuous semifinite trace, and π a faithful representation of \mathcal{A} . Let us denote by $\mathcal{R}_\pi(\mathcal{A}, \tau)$ the linear span of the separating elements for the \mathcal{R} -cuts (w.r.t. τ) in $\pi(\mathcal{A})$. When π is the universal representation, we denote it simply by $\mathcal{R}(\mathcal{A}, \tau)$, and call it the *enveloping Riemann algebra* of \mathcal{A} . For the sake of convenience we use the shorthand notation $\mathcal{A}^{\mathcal{R}} \equiv \mathcal{R}_\pi(\mathcal{A}, \tau)$, when π and τ are clear from the context. The C^* -algebra \mathcal{A} is called a *Riemann algebra* if it contains all the separating elements of its \mathcal{R} -cuts.

If $x \in \mathcal{A}_{\text{sa}}^{\mathcal{R}}$ and (A^-, A^+) is a corresponding \mathcal{R} -cut, either all or none of the a 's in A^\pm have finite trace. In the first case we set $\tau(x) := \inf \tau(a^+) \equiv \sup \tau(a^-)$. In the second case, and if $x \geq 0$, we set $\tau(x) := +\infty$. This is the unique positive functional extending τ to $\mathcal{A}^{\mathcal{R}}$.

The first property of a Riemann algebra \mathcal{A} is that it is closed under Riemann functional calculus. Moreover, the elements of \mathcal{A} with τ -finite support are separating elements between upper and lower Riemann sums made up with projections in \mathcal{A} .

LEMMA 2.2. Let (\mathcal{A}, τ) be a C^* -algebra with a tracial weight, and denote by $\mathcal{J}(\tau)$ its domain. Then, given a selfadjoint element x in \mathcal{A} ,

$$\tau(\phi_\varepsilon(|x|)) < \infty, \quad \forall \varepsilon > 0 \Leftrightarrow x \in \overline{\mathcal{J}(\tau)},$$

where, for any $\varepsilon > 0$, ϕ_ε is an increasing continuous function from \mathbb{R}_+ to \mathbb{R} such that $\phi_\varepsilon = 0$ on $[0, \varepsilon/2]$ and $\phi_\varepsilon = 1$ on $(\varepsilon, +\infty)$.

Proof. (\Rightarrow) We have $|\tau(x\phi_\varepsilon(|x|))| \leq \|x\| \tau(\phi_\varepsilon(|x|)) < \infty$, i.e., $x\phi_\varepsilon(|x|) \in \mathcal{J}(\tau)$, and $\|x\phi_\varepsilon(|x|) - x\| < \varepsilon$, which implies $x \in \overline{\mathcal{J}(\tau)}$.

(\Leftarrow) By definition [18, p. 175], $|x| \phi_\varepsilon(|x|)$ belongs to the Pedersen ideal $K(\overline{\mathcal{J}(\tau)})$. Since $\mathcal{J}(\tau)$ is a dense ideal in $\overline{\mathcal{J}(\tau)}$, $K(\overline{\mathcal{J}(\tau)}) \subset \mathcal{J}(\tau)$ by minimality, hence $\tau(|x| \phi_\varepsilon(|x|)) < \infty$. Then $\tau(\phi_\varepsilon(|x|)) < \frac{2}{\varepsilon} \tau(|x| \phi_\varepsilon(|x|)) < \infty$. ■

Let us consider, for any selfadjoint x , the measure μ_x on $\sigma(x) \setminus \{0\}$ defined by

$$\int f(\lambda) d\mu_x(\lambda) = \tau(f(x)), \quad f \in C_0(\sigma(x) \setminus \{0\}). \quad (2.2)$$

PROPOSITION 2.3. *Let \mathcal{A} be a Riemann algebra w.r.t. a semicontinuous semi-finite trace τ . Then:*

- (i) \mathcal{A} is closed under Riemann functional calculus, namely

$$\forall x \in \mathcal{A}, \quad f \in \mathcal{R}_0(\sigma(x) \setminus \{0\}, \mu_x) \Rightarrow f(x) \in \mathcal{A},$$

where $\mathcal{R}_0(X, \mu)$ denotes the set of Riemann measurable functions on X w.r.t. μ and vanishing at infinity (cf. Appendix). In particular Riemann measurable spectral projections of selfadjoint elements of \mathcal{A} belong to \mathcal{A} , where a spectral projection $e_\Omega(x)$ is Riemann measurable if $\Omega \subset \subset \sigma(x) \setminus \{0\}$ and $\mu_x(\partial\Omega) = 0$.

- (ii) Let $x \in \mathcal{A}_{\text{sa}}$ have τ -finite support. Then x is the separating element between \mathcal{R} -cuts made by linear combinations of projections in \mathcal{A} . In particular, if τ is densely defined on \mathcal{A} , then \mathcal{A} is generated by its projections as a Riemann algebra.

Proof. (i) It is sufficient to prove the assertion for a real-valued function f . Then, by Proposition 6.1 in the Appendix, there exists an open set $V \subset \sigma(x) \setminus \{0\}$, with $\mu_x(V)$ finite, and functions $h, h_\varepsilon^\pm \in C_0(\sigma(x) \setminus \{0\})$, with h_ε^\pm vanishing outside V , such that $\int (h_\varepsilon^+ - h_\varepsilon^-) d\mu_x < \varepsilon$ and $h_\varepsilon^- \leq f - h \leq h_\varepsilon^+$. Then $h(x), h_\varepsilon^\pm(x) \in \mathcal{A}$ by continuous functional calculus, and $f(x) - h(x) \in \mathcal{A}$ because it is a Riemann algebra.

(ii) Since x has τ -finite support, the measure μ_x is finite, therefore the set $S(x) = \{\alpha \in \sigma(x) : \mu_x(\alpha) \neq 0\}$ is at most countable. Consider the separated classes given by upper and lower Riemann sums of the function $f(t) = t$ on $\sigma(x)$, corresponding to subdivisions which do not intersect $S(x)$. Such classes are μ_x -contiguous and the corresponding spectral projections belong to \mathcal{A} by (i). The corresponding functional calculi of x give the

\mathcal{R} -cut for x . Concerning the last statement, we observe that $x\phi_\varepsilon(|x|)$ has τ -finite support for any $\varepsilon > 0$ and any $x \in \mathcal{A}_{\text{sa}}$, by Lemma 2.2. Then the thesis easily follows by part (i). ■

Then we may state the main theorem of this section.

THEOREM 2.4. *Let \mathcal{A} be a C^* -algebra with a semicontinuous semifinite trace τ , acting on a Hilbert space \mathcal{H} . Then*

- (i) $\mathcal{A}^{\mathcal{R}}$ is a C^* -algebra
- (ii) the above described extension of τ to $\mathcal{A}^{\mathcal{R}}$ is a trace, and $(\mathcal{A}^{\mathcal{R}}, \tau)$ is a Riemann algebra
- (iii) the GNS representation π_τ of \mathcal{A} extends to a representation ρ_π of $\mathcal{A}^{\mathcal{R}}$ into $\pi_\tau(\mathcal{A})''$, and $\tau|_{\mathcal{A}^{\mathcal{R}}}$ may be identified with the pull-back of the trace on $\pi_\tau(\mathcal{A})''$ via ρ_π . As a consequence τ is semicontinuous semifinite on $\mathcal{A}^{\mathcal{R}}$.

The rest of this section is devoted to the proof of Theorem 2.4. In order to do that we have to introduce *a priori* different kinds of Dedekind closures of (\mathcal{A}, τ) , namely we shall consider unbounded cuts (\mathcal{U} -cuts), where the uniform boundedness property is removed, and the corresponding \mathcal{U} -closure $\mathcal{A}^{\mathcal{U}}$, and tight cuts (\mathcal{T} -cuts), where the uniform boundedness is strengthened by requiring that the a_ε^\pm 's in (2.1) verify

$$\sup \sigma(a_\varepsilon^+) < \sup \sigma(x) + \varepsilon, \quad \inf \sigma(a_\varepsilon^-) > \inf \sigma(x) - \varepsilon,$$

and the corresponding \mathcal{T} -closure, $\mathcal{A}^{\mathcal{T}}$. Of course we have $\mathcal{A} \subset \mathcal{A}^{\mathcal{T}} \subset \mathcal{A}^{\mathcal{R}} \subset \mathcal{A}^{\mathcal{U}}$, and τ extends to $\mathcal{A}^{\mathcal{U}}$ as well. We shall see that $\mathcal{A}^{\mathcal{T}} = \mathcal{A}^{\mathcal{R}}$ is a C^* -algebra, while $\mathcal{A}^{\mathcal{U}}$ is not (cf. Example 2.17 below). It becomes a C^* -algebra iff it coincides with $\mathcal{A}^{\mathcal{R}}$. This fact may be seen as a motivation for choosing $\mathcal{A}^{\mathcal{R}}$ instead of $\mathcal{A}^{\mathcal{U}}$ as the family of Riemann measurable elements. As we shall see in Section 3, the possibility of taking products inside the set of Riemann measurable elements is crucial for constructing $\overline{\mathcal{A}^{\mathcal{R}}}$.

Remark 2.5. In the abelian case, \mathcal{U} -closure, \mathcal{R} -closure and \mathcal{T} -closure coincide, indeed in this case we may always find a very tight cut, for which $\sup \sigma(a_\varepsilon^+) = \sup \sigma(x)$ and $\inf \sigma(a_\varepsilon^-) = \inf \sigma(x)$. We conjecture that this is not always the case in the nonabelian setting.

LEMMA 2.6. *The sets of separating elements for \mathcal{R} -cuts and \mathcal{U} -cuts are the selfadjoint parts of the $*$ -bimodules $\mathcal{A}^{\mathcal{R}}$ and $\mathcal{A}^{\mathcal{U}}$ on \mathcal{A} .*

Proof. Linearity and $*$ -invariance are obvious. Now let $a \in \mathcal{A}$ and $x \in \mathcal{A}_{sa}''$ a separating element for the \mathcal{R} -cut $\{a_\varepsilon^\pm\}$. Then $a^*a_\varepsilon^\pm a$ gives an \mathcal{R} -cut for a^*xa . From the equalities

$$xy + yx = (x+1)^* y(x+1) - xyx - y \quad (2.3)$$

$$i(xy - yx) = (x+i)^* y(x+i) - xyx - y \quad (2.4)$$

which hold for any pair of selfadjoint elements we then get the bimodule property for $\mathcal{A}^{\mathcal{R}}$. The proof for $\mathcal{A}^{\mathcal{U}}$ is analogous. ■

LEMMA 2.7. *There is a unique linear positive extension $\rho: \mathcal{A}^{\mathcal{U}} \rightarrow \pi_\tau(\mathcal{A})''$ of the GNS representation of \mathcal{A} .*

Proof. Let $x \in \mathcal{A}_{sa}^{\mathcal{U}}$, and $\{a_\varepsilon^\pm\}$ be a \mathcal{U} -cut for x . Then $\pi_\tau(a_\varepsilon^\pm)$ is a \mathcal{U} -cut, and, $\pi_\tau(a_\varepsilon^+)$ and $\pi_\tau(a_\varepsilon^-)$ converge to the same element in $L^1(\pi_\tau(\mathcal{A})'', \tau)$, which is the unique separating element of the \mathcal{U} -cut, hence belongs to $\pi_\tau(\mathcal{A})''$. Setting

$$\rho(x) := \lim_{\varepsilon \rightarrow 0} \pi_\tau(a_\varepsilon^+) = \lim_{\varepsilon \rightarrow 0} \pi_\tau(a_\varepsilon^-), \quad (2.5)$$

it follows easily that $\rho(x)$ does not depend on the \mathcal{U} -cut and ρ is linear and positive. ■

LEMMA 2.8. (i) ρ is a bimodule map.

(ii) $\tau|_{\mathcal{A}^{\mathcal{U}}} = \tau \circ \rho$, hence it is a trace on $\mathcal{A}^{\mathcal{U}}$ as a bimodule on \mathcal{A} .

Proof. (i) This follows easily from Eqs. (2.3), (2.4).

(ii) This follows from positivity of ρ and τ . ■

LEMMA 2.9. *The \mathcal{U} -closure of \mathcal{A} is \mathcal{U} -closed, namely the set of separating elements in \mathcal{A}_{sa}'' for \mathcal{U} -cuts in $\mathcal{A}^{\mathcal{U}}$ is contained in $\mathcal{A}^{\mathcal{U}}$.*

Proof. Indeed if $\{x_{\varepsilon'}^\pm\}_{\varepsilon' > 0}$ is a \mathcal{U} -cut in $\mathcal{A}^{\mathcal{U}}$ for $x \in \mathcal{A}_{sa}''$, and $\{a(x_{\varepsilon'}^\pm)_\varepsilon^\pm\}_{\varepsilon > 0}$ is a \mathcal{U} -cut in \mathcal{A} for $x_{\varepsilon'}^\pm$, then $\{(a(x_{\varepsilon'}^+)_\varepsilon^+, a(x_{\varepsilon'}^-)_\varepsilon^-)\}$ gives a \mathcal{U} -cut in \mathcal{A} for x . ■

With the above terminology, Theorem 2.4 shows that the \mathcal{R} -closure of a C^* -algebra is \mathcal{R} -closed.

LEMMA 2.10. *Let \mathcal{A} be a C^* -algebra acting on a Hilbert space \mathcal{H} , with a semi-continuous semifinite trace τ . If $C_0(x) := \{f(x) : f \in C_0(\sigma(x) \setminus \{0\})\}$ is contained in $\mathcal{A}^{\mathcal{U}}$ for a selfadjoint element x , then $C_0(x) \subset \mathcal{A}^{\mathcal{T}}$. In particular*

$\mathcal{A}^{\mathcal{T}}$, $\mathcal{A}^{\mathcal{R}}$ and $\mathcal{A}^{\mathcal{U}}$ have the same projections, and any C^* -subalgebra of $\mathcal{A}^{\mathcal{U}}$ is actually contained in $\mathcal{A}^{\mathcal{T}}$.

Proof. First we observe that any projection e in $\mathcal{A}^{\mathcal{U}}$ belongs to $\mathcal{A}^{\mathcal{T}}$. For any $\delta > 0$ we consider the operator increasing functions (cf. [18])

$$f_{\delta}^{+}(z) = \frac{(1+\delta)z}{\delta+z}; \quad f_{\delta}^{-}(z) = \frac{\delta z}{1+\delta-z}.$$

If $\{a_{\varepsilon}^{\pm}\}_{\varepsilon>0}$ gives a \mathcal{U} -cut for e , $\{f_{\delta}^{\pm}(a_{\varepsilon}^{\pm})\}_{\varepsilon>0}$ gives an \mathcal{R} -cut for any $\delta > 0$, indeed $f_{\delta}^{\pm}(e) = e$ together with operator monotonicity imply that $f_{\delta}^{-}(a_{\varepsilon}^{-}) \leq e \leq f_{\delta}^{+}(a_{\varepsilon}^{+})$, and we have the estimate

$$\begin{aligned} \tau(f_{\delta}^{+}(a_{\varepsilon}^{+}) - f_{\delta}^{-}(a_{\varepsilon}^{-})) &= \tau(f_{\delta}^{+}(a_{\varepsilon}^{+}) - f_{\delta}^{+}(e)) + \tau(f_{\delta}^{-}(e) - f_{\delta}^{-}(a_{\varepsilon}^{-})) \\ &= (1+\delta) \delta [\tau((a_{\varepsilon}^{+} + \delta)^{-1} (a_{\varepsilon}^{+} - e)(e + \delta)^{-1}) \\ &\quad + \tau((1 + \delta - e)^{-1} (e - a_{\varepsilon}^{-})(1 + \delta - a_{\varepsilon}^{-})^{-1})] \\ &\leq \frac{1+\delta}{\delta} \varepsilon. \end{aligned}$$

Moreover, we have $\sup \sigma(f_{\delta}^{+}(a_{\varepsilon}^{+})) \leq 1 + \delta$ and $\inf \sigma(f_{\delta}^{-}(a_{\varepsilon}^{-})) \geq -\delta$, therefore we may easily extract a \mathcal{T} -cut for e .

Now let $C_0(x) \subset \mathcal{U}$; we want to show that $x \in \mathcal{A}^{\mathcal{T}}$. First observe that $C_0(x^{\pm}) \subset \mathcal{U}$, where x^{\pm} is the positive, resp. negative part of x . If 0 is in the convex hull of the spectrum of x , then $\inf \sigma(x^{\pm}) = 0$, hence \mathcal{T} -cuts for x^{\pm} give a \mathcal{T} -cut for x , namely we may restrict to the case $\inf \sigma(x) = 0$. If $\inf \sigma(x) = m > 0$, then $I \in \mathcal{A}^{\mathcal{U}}$ by hypothesis, and it is sufficient to prove the statement for $x - mI$, while if $\sup \sigma(x) = m < 0$, it is sufficient to prove the statement for $-x + mI$, therefore again we may suppose $\inf \sigma(x) = 0$. Hence, by multiplying with an appropriate constant, we may assume $\inf \sigma(x) = 0$ and $\sup \sigma(x) = 1$.

Then we fix a $\delta > 0$ and apply Lemma 6.3 in the Appendix to the identity function on the spectrum of x . We may therefore find a sequence e_n of Riemann measurable spectral projections of x and a sequence of positive numbers α_n such that $\sum_n \alpha_n e_n = x$ and $\sum_n \alpha_n = 1$.

Moreover, since $\mathcal{A}^{\mathcal{U}}$ is \mathcal{U} -closed by Proposition 2.9, all the Riemann measurable spectral projections of x belong to $\mathcal{A}^{\mathcal{U}}$ (cf. the proof of Proposition 2.3).

Therefore, for any e_n we obtain an \mathcal{R} -cut $(\{a_{n,\varepsilon}^{-}\}_{\varepsilon>0}, \{a_{n,\varepsilon}^{+}\}_{\varepsilon>0})$ such that $\sup_{\varepsilon} \sigma(a_{n,\varepsilon}^{+}) \leq 1 + \delta$ and $\inf_{\varepsilon} \sigma(a_{n,\varepsilon}^{-}) \geq -\delta$. Hence, setting $a_{\varepsilon}^{\pm} = \sum_n \alpha_n a_{n,\varepsilon}^{\pm}$, we obtain

$$a_{\varepsilon}^{-} \leq x \leq a_{\varepsilon}^{+},$$

$$\tau(a_{\varepsilon}^{+} - a_{\varepsilon}^{-}) \leq \sum_n \alpha_n \tau(a_{n,\varepsilon}^{+} - a_{n,\varepsilon}^{-}) \leq \varepsilon,$$

$$\sup \sigma(a_{\varepsilon}^{+}) \leq \sum_n \alpha_n \sup \sigma(a_{n,\varepsilon}^{+}) \leq 1 + \delta,$$

$$\inf \sigma(a_{\varepsilon}^{-}) \geq \sum_n \alpha_n \inf \sigma(a_{n,\varepsilon}^{-}) \geq -\delta.$$

From these \mathcal{R} -cuts it is easy to extract a \mathcal{T} -cut. Finally, we observe that $C_0(f(x)) \subset \mathcal{A}^{\mathcal{U}}$ for any $f \in C_0(\sigma(x) \setminus \{0\})$, hence $C_0(x) \subset \mathcal{A}^{\mathcal{T}}$. ■

LEMMA 2.11. *Let $X \subset \mathcal{A}^{\mathcal{T}}$ be a $*$ -invariant vector space. Then $\bar{X} \subset \mathcal{A}^{\mathcal{R}}$.*

Proof. Let $x \in \bar{X}_{\text{sa}}$, $x_n \rightarrow x$ in norm, $x_n \in X_{\text{sa}}$. First, possibly passing to a subsequence, we may assume $\|x - x_n\| < 2^{-n}$. Then set $y_n = x_{n+1} - x_n \in X$ and observe that $\|y_n\| < 3 \cdot 2^{-(n+1)}$. By hypothesis we may find a \mathcal{T} -cut in \mathcal{A} for y_n , hence elements $a_{n\varepsilon}^{\pm} \in \mathcal{A}$ such that $a_{n\varepsilon}^{-} \leq y_n \leq a_{n\varepsilon}^{+}$, $\|a_{n,\varepsilon}^{\pm}\| \leq 2 \cdot 2^{-n}$, $\tau(a_{n\varepsilon}^{+} - a_{n\varepsilon}^{-}) \leq \varepsilon 2^{-n}$. Then $a_{\varepsilon}^{\pm} := \sum_n a_{n\varepsilon}^{\pm}$ belongs to \mathcal{A} by uniform convergence, and

$$\tau(a_{\varepsilon}^{+} - a_{\varepsilon}^{-}) \leq \sum_n \tau(a_{n\varepsilon}^{+} - a_{n\varepsilon}^{-}) \leq \varepsilon,$$

where the first inequality follows by the semicontinuity of τ . Since $\|a_{\varepsilon}^{\pm}\| \leq \sum_n \|a_{n\varepsilon}^{\pm}\| \leq 2$, we get $x \in \mathcal{A}^{\mathcal{R}}$. ■

LEMMA 2.12. *Let $x \in \mathcal{A}^{\mathcal{R}}$, and consider the functions $f_t(z) := \frac{tz}{1-tz}$, $t \in [0, 1)$. Then $f_t(x) \in \mathcal{A}^{\mathcal{R}}$, and $\rho(f_t(x)) = f_t(\rho(x))$, for sufficiently small t .*

Proof. Let x be a separating element for the \mathcal{R} -cut $\{a_{\varepsilon}^{\pm}\} \subset \mathcal{A}$, with $r = \sup_{\varepsilon} \|a_{\varepsilon}^{\pm}\|$. Then $f_t(a_{\varepsilon}^{\pm}) \in \mathcal{A}$ for any $t \in [0, 1/r)$. Since, for any such t , the function f_t is operator monotone increasing (cf. [18]) on $(-\infty, r)$, we get

$$f_t(a_{\varepsilon}^{-}) \leq f_t(x) \leq f_t(a_{\varepsilon}^{+}), \quad t \in [0, 1/r)$$

and

$$\begin{aligned} f_t(a_{\varepsilon}^{+}) - f_t(a_{\varepsilon}^{-}) &= (I - tx_{\varepsilon}^{+})^{-1} (tx_{\varepsilon}^{+}(I - tx_{\varepsilon}^{-}) - (I - tx_{\varepsilon}^{+})tx_{\varepsilon}^{-})(I - tx_{\varepsilon}^{-})^{-1} \\ &= t(I - tx_{\varepsilon}^{+})^{-1} (x_{\varepsilon}^{+} - x_{\varepsilon}^{-})(I - tx_{\varepsilon}^{-})^{-1}. \end{aligned}$$

Then, taking $0 \leq t < \frac{1}{2r}$, we get

$$\|f_t(a_\varepsilon^\pm)\| \leq 1$$

$$\tau(f_t(a_\varepsilon^+) - f_t(a_\varepsilon^-)) \leq \frac{2\varepsilon}{r},$$

which means that $f_t(x)$ is a separating element for the \mathcal{R} -cut $\{f_t(a_\varepsilon^\pm)\}$, therefore, $f_t(x)$ belongs to $\mathcal{A}^{\mathcal{R}}$ for any $t \in [0, \frac{1}{2r})$. Finally, making use of Eq. (2.5) (where the limits are in L^1) we get

$$\rho(f_t(x)) = \lim_{\varepsilon \rightarrow 0} \pi_\tau(f_t(a_\varepsilon^\pm)) = \lim_{\varepsilon \rightarrow 0} f_t(\pi_\tau(a_\varepsilon^\pm)) = f_t(\rho(x)). \quad \blacksquare$$

LEMMA 2.13. *Let \mathcal{A} be a C^* -algebra acting on a Hilbert space \mathcal{H} , with a semi-continuous semifinite trace τ . If the norm closure of $\mathcal{A}^{\mathcal{R}}$ is contained in $\mathcal{A}^{\mathcal{U}}$, then $\mathcal{A}^{\mathcal{T}} = \mathcal{A}^{\mathcal{R}}$, and Theorem 2.4 holds for \mathcal{A} .*

Proof. The proof requires some intermediate steps. Some of the arguments are taken from the proof of Kadison [13] that the Borel closure of a C^* -algebra is a C^* -algebra.

Step (i). $\mathcal{A}^{\mathcal{T}} = \mathcal{A}^{\mathcal{R}}$ is a Banach space and is closed under continuous functional calculus.

Let $x \in \mathcal{A}^{\mathcal{R}}$, then by Lemma 2.12 we have $f_t(x) \in \mathcal{A}^{\mathcal{R}}$, and the equality

$$x^n = \lim_{t \rightarrow 0} \left(f_t(x) - \sum_{k=1}^{n-1} t^k x^k \right) t^{-n}, \quad (2.6)$$

inductively implies $x^n \in \mathcal{A}^{\mathcal{U}}$, hence $C_0(x) \subset \mathcal{A}^{\mathcal{U}}$. Then Lemma 2.10 shows that $C_0(x) \subset \mathcal{A}^{\mathcal{T}}$, namely $\mathcal{A}^{\mathcal{T}} = \mathcal{A}^{\mathcal{R}}$ and it is closed under continuous functional calculus. Then, since $\mathcal{A}^{\mathcal{R}}$ is a vector space, we may apply Lemma 2.11 to $\mathcal{A}^{\mathcal{T}}$, obtaining that $\mathcal{A}^{\mathcal{R}}$ is norm closed.

Step (ii). $\mathcal{A}^{\mathcal{R}}$ is \mathcal{R} -closed, namely if $x \in \mathcal{A}^{\mathcal{U}}$ is a separating element for an \mathcal{R} -cut $\{x_\varepsilon^\pm\} \subset \mathcal{A}^{\mathcal{R}}$, then $x \in \mathcal{A}^{\mathcal{R}}$.

Choose a \mathcal{T} -cut $\{(a_\varepsilon^\pm)_\delta^\pm\} \subset \mathcal{A}$ for x_ε^\pm . Then $\{(a_\varepsilon^-)_\varepsilon^-, (a_\varepsilon^+)_\varepsilon^+\}$ is an \mathcal{R} -cut for x , and $x \in \mathcal{A}^{\mathcal{R}}$.

Step (iii). ρ is norm-continuous and $\rho(x^2) = \rho(x)^2$.

Indeed, adjoining the identity to \mathcal{A} if necessary, if $x \in \mathcal{A}_{\text{sa}}^{\mathcal{R}}$, then, as ρ is a linear and positive map, we get $\|\rho(x)\| \leq \|x\| \|\pi(1)\|$, and by linearity of ρ we get norm continuity. The last equation follows by applying ρ to Eq. (2.6) for $n=2$ and using Lemma 2.12.

Step (iv). If $x, y \in \mathcal{A}_{\text{sa}}^{\mathcal{R}}$, then $xy + yx \in \mathcal{A}^{\mathcal{R}}$ and $\rho(xy + yx) = \rho(x)\rho(y) + \rho(y)\rho(x)$.

This is immediate by steps (i) and (iii), and

$$xy + yx = (x + y)^2 - x^2 - y^2. \quad (2.7)$$

Step (v). If $x, y \in \mathcal{A}_{\text{sa}}^{\mathcal{R}}$, then $xyx \in \mathcal{A}^{\mathcal{R}}$ and $\rho(xyx) = \rho(x) \rho(y) \rho(x)$. This is immediate by step (iv) and

$$2xyx = (xy + yx)x + x(xy + yx) - (yx^2 + x^2y). \quad (2.8)$$

Step (vi). If $x \in \mathcal{A}_{\text{sa}}^{\mathcal{R}}, y \in \mathcal{A}_{\text{sa}}$ then $(x+i)^* y(x+i) \in \mathcal{A}^{\mathcal{R}}$ and $\rho((x+i)^* y(x+i)) = (\rho(x) + i)^* \rho(y)(\rho(x) + i)$.

This follows by Eqs. (2.4) and (2.8) for $x \in \mathcal{A}_{\text{sa}}^{\mathcal{R}}, y \in \mathcal{A}_{\text{sa}}$, and by the bimodule property.

Step (vii). If $x, y \in \mathcal{A}_{\text{sa}}^{\mathcal{R}}$, then $(x+i)^* y(x+i) \in \mathcal{A}^{\mathcal{R}}$ and $\rho((x+i)^* y(x+i)) = (\rho(x) + i)^* \rho(y)(\rho(x) + i)$.

Let $\{b_{\varepsilon}^{\pm}\}$ be an \mathcal{R} -cut for y ; then we have

$$(x+i)^* b_{\varepsilon}^{-}(x+i) \leq (x+i)^* y(x+i) \leq (x+i)^* b_{\varepsilon}^{+}(x+i),$$

and, by step (vi),

$$\begin{aligned} \tau((x+i)^* (b_{\varepsilon}^{+} - b_{\varepsilon}^{-})(x+i)) &= \tau \circ \rho((x+i)^* (b_{\varepsilon}^{+} - b_{\varepsilon}^{-})(x+i)) \\ &\leq \|x+i\|^2 \tau(b_{\varepsilon}^{+} - b_{\varepsilon}^{-}) \leq \varepsilon \|x+i\|^2 \end{aligned}$$

so that $\{(x+i)^* b_{\varepsilon}^{\pm}(x+i)\} \subset \mathcal{A}^{\mathcal{R}}$ is an \mathcal{R} -cut for $(x+i)^* y(x+i)$. Then the thesis follows by step (ii) and Eq. (2.5).

Step (viii). $\mathcal{A}^{\mathcal{R}}$ is a C*-algebra and ρ is a representation.

The only missing property is multiplicativity, which directly follows from steps (iv) and (vii), and Eq. (2.4).

Step (ix). ρ is the GNS representation of $\mathcal{A}^{\mathcal{R}}$ w.r.t. τ .

Let us denote by $(\pi_{\tau}, \mathcal{H}_{\tau}, \eta_{\tau})$ the GNS triple of \mathcal{A} w.r.t. τ . Set $\mathfrak{N} := \{x \in \mathcal{A}^{\mathcal{R}} : \tau(x^*x) < \infty\}$, and define the linear map $\eta: \mathfrak{N} \rightarrow \mathcal{H}_{\tau}$ as follows. If $x \in \mathfrak{N}_{+}$, and $\{a_{\varepsilon}\} \subset \mathfrak{N} \cap \mathcal{A}_{+}$ is s.t. $x \leq a_{\varepsilon}$ and $\tau(|a_{\varepsilon} - x|^2) < \varepsilon$, let us set $\eta(x) := \lim_{\varepsilon \rightarrow 0} \eta_{\tau}(a_{\varepsilon})$, where the limit is independent of $\{a_{\varepsilon}\}$, and extend by linearity to all of \mathfrak{N} , which is generated by its positive elements. Then it is easy to see that $(\eta(x), \eta(y)) = \tau(x^*y)$, for $x, y \in \mathfrak{N}$, and all that remains to show is that $\rho(x) \eta(y) = \eta(xy)$ for $x \in \mathcal{A}^{\mathcal{R}}, y \in \mathfrak{N}$, and it suffices to show it for $x, y \geq 0$. Let us prove it first for $y \in \mathfrak{N} \cap \mathcal{A}_{+}$. Observe that, using Eqs. (2.3) and (2.4), we can write

$$\begin{aligned}
xy &= \frac{1}{2} (y+1) x(y+1) + \frac{i}{2} (y+i)^* x(y+i) - \frac{1+i}{2} yxy - \frac{1+i}{2} x \\
&\equiv \sum_{k=1}^4 \lambda_k b_k^* x b_k,
\end{aligned} \tag{2.9}$$

where $\lambda_k \in \mathbb{C}$, $b_k \in \mathcal{A} + \mathbb{C}$. Let now $\{a_\varepsilon^\pm\} \subset \mathcal{A}$ be an \mathcal{R} -cut, with x as separating element. Then, using Eq. (2.9), we get $a_\varepsilon^+ y = \sum_{k=1}^4 \lambda_k b_k^* a_\varepsilon^+ b_k$, so that $b_k^* x b_k \leq b_k^* a_\varepsilon^+ b_k$, and

$$\tau(|b_k^*(a_\varepsilon^+ - x) b_k|^4) \leq \|b_k\|^2 \|a_\varepsilon^+ - x\| \tau(a_\varepsilon^+ - x) \rightarrow 0, \quad \varepsilon \rightarrow 0,$$

hence,

$$\begin{aligned}
\lim_{\varepsilon \rightarrow 0} \pi_\tau(a_\varepsilon^+) \eta_\tau(y) &= \lim_{\varepsilon \rightarrow 0} \eta_\tau(a_\varepsilon^+ y) = \lim_{\varepsilon \rightarrow 0} \sum_{k=1}^4 \lambda_k \eta(b_k^* a_\varepsilon^+ b_k) \\
&= \sum_{k=1}^4 \lambda_k \eta(b_k^* x b_k) = \eta(xy).
\end{aligned}$$

But we also have

$$\|[\pi_\tau(a_\varepsilon^+) - \rho(x)] \eta_\tau(y)\|^2 \leq \|\pi_\tau(a_\varepsilon^+) - \rho(x)\| \|\pi_\tau(a_\varepsilon^+) - \rho(x)\|_1 \|y\|^2 \rightarrow 0,$$

as $\varepsilon \rightarrow 0$, so that $\rho(x) \eta(y) = \eta(xy)$. Finally for $x \in \mathcal{A}_+^{\mathcal{R}}$, $y \in \mathfrak{N}_+$, $a \in \mathfrak{N} \cap \mathcal{A}_+$, we get

$$\begin{aligned}
(\eta(a), \rho(x) \eta(y)) &= (\rho(x) \eta(a), \eta(y)) = (\eta(xa), \eta(y)) \\
&= \tau(axy) = (\eta(a), \eta(xy))
\end{aligned}$$

and the thesis follows. ■

In the following subsections we shall prove that the closure of $\mathcal{A}^{\mathcal{R}}$ is contained in $\mathcal{A}^{\mathcal{U}}$ in the cases in which \mathcal{A} is unital and the trace is finite, \mathcal{A} is non-unital and the trace is densely defined, and finally in the non-densely defined case, thus completing the proof of Theorem 2.4.

COROLLARY 2.14. *$\mathcal{A}^{\mathcal{R}}$ is the unique Riemann algebra between \mathcal{A} and $\mathcal{A}^{\mathcal{U}}$. $\mathcal{A}^{\mathcal{U}}$ is a C^* -algebra if and only if it coincides with $\mathcal{A}^{\mathcal{R}}$.*

Proof. By Lemma 2.13 and Theorem 2.4 we obtain that $\mathcal{A}^{\mathcal{R}}$, being the minimal Riemann algebra between \mathcal{A} and $\mathcal{A}^{\mathcal{U}}$ and being the maximal C^* -algebra between \mathcal{A} and $\mathcal{A}^{\mathcal{U}}$, is indeed the unique Riemann algebra between \mathcal{A} and $\mathcal{A}^{\mathcal{U}}$. As a consequence, either $\mathcal{A}^{\mathcal{R}} = \mathcal{A}^{\mathcal{U}}$, or $\mathcal{A}^{\mathcal{U}}$ is not a C^* -algebra. ■

Remark 2.15. We say that a linear subspace X in \mathcal{A}^{**} is completely τ -measurable if, for any faithful representation π of \mathcal{A} , the map $\rho_\pi := \pi_\tau \cdot \pi^{-1}: \pi(\mathcal{A}) \rightarrow \pi_\tau(\mathcal{A})$ extends to a linear map from $\pi(X)$ in such a way that the following diagram is commutative,

$$\begin{array}{ccc} X & \xrightarrow{\quad} & \mathcal{A}^{**} \\ \pi \downarrow & & \downarrow \pi_\tau \\ \pi(X) & \xrightarrow{\rho_\pi} & \pi_\tau(\mathcal{A})'' \end{array}$$

When π is not normal, τ cannot be extended to $\pi(\mathcal{A})''$. However, if \mathcal{R} is a completely measurable C*-algebra, $\mathcal{A} \subset \mathcal{R} \subset \mathcal{A}^{**}$, τ uniquely extends to a trace on $\pi(\mathcal{R})$. One can show that $\mathcal{U}(\mathcal{A}, \tau)$, hence $\mathcal{R}(\mathcal{A}, \tau)$, are completely τ -measurable, however the algebra generated by $\mathcal{U}(\mathcal{A}, \tau)$ may fail to be completely measurable. Observe that $\pi(\mathcal{U}(\mathcal{A}, \tau)) \subset \mathcal{A}^{\mathcal{U}}$ and $\pi(\mathcal{R}(\mathcal{A}, \tau)) \subset \mathcal{A}^{\mathcal{R}}$, whereas it is not known to the authors if equalities hold.

2.1. Finite Trace on a Unital C*-Algebra

In this subsection we prove Theorem 2.4 in the case of a unital C*-algebra with a finite trace. This will be an immediate corollary of the following theorem.

THEOREM 2.16. *Let (\mathcal{A}, τ) be a unital C*-algebra acting on a Hilbert space \mathcal{H} , τ a tracial state on \mathcal{A} , then $\mathcal{A}^{\mathcal{U}}$ is norm closed; therefore Theorem 2.4 holds for \mathcal{A} .*

Proof. Let $x \in \overline{\mathcal{A}^{\mathcal{U}}}$ and $x_n \in \mathcal{A}^{\mathcal{U}}$, $\|x_n - x\| < \frac{1}{n}$. Then x is a separating element for the \mathcal{U} -cut $(\{x_n - \frac{1}{n}\}, \{x_n + \frac{1}{n}\})$, so that $x \in \mathcal{A}^{\mathcal{U}}$ since $\mathcal{A}^{\mathcal{U}}$ is \mathcal{U} -closed. Then the result follows by Lemma 2.13. ■

Since $\mathcal{A}^{\mathcal{U}}$ is a norm closed *-bimodule, $\mathcal{A}^{\mathcal{R}} \neq \mathcal{A}^{\mathcal{U}} \Leftrightarrow \mathcal{A}^{\mathcal{U}}$ is not an algebra. We conclude this subsection with an example showing that this is indeed possible. Apparently such phenomenon depends on τ being non-faithful on $\mathcal{A}^{\mathcal{R}}$, which can happen even if τ is faithful on \mathcal{A} .

EXAMPLE 2.17. Let μ be the sum of the Lebesgue measure on $[0, 1]$ plus the Dirac measure in $\{0\}$, and consider the C*-algebra $\mathcal{A} = \{f \in C([0, 1]) \otimes M(2) : f_{12}(0) = f_{21}(0) = 0\}$ acting on the Hilbert space $\mathcal{H} = L^2([0, 1], d\mu) \otimes \mathbb{C}^2$. Let τ be the state on $\mathcal{B}(\mathcal{H})$ defined by

$$\tau(f) := \frac{1}{2} \int \operatorname{tr} f(t) dt + \frac{1}{2} f_{11}(0).$$

Then τ is a trace on \mathcal{A}'' and is faithful on \mathcal{A} , $\mathcal{A}^{\mathcal{R}}$ is given by the $M(2)$ -valued Riemann measurable functions f on $[0, 1]$ s.t. f_{11} is continuous in 0 and f_{12}, f_{21} vanish and are continuous in 0, $\mathcal{A}^{\mathcal{U}}$ is given by the $M(2)$ -valued Riemann measurable functions f on $[0, 1]$ s.t. f_{11} is continuous in 0 and f_{12}, f_{21} vanish in 0. In particular $\mathcal{A}^{\mathcal{U}}$ is not an algebra.

Proof. Since \mathcal{A}'' is given by $\{f \in L^\infty([0, 1], d\mu) \otimes M(2) : f_{12}(0) = f_{21}(0) = 0\}$, the trace property follows.

Since elements in $\mathcal{A}^{\mathcal{U}}$ are separating elements for \mathcal{U} -cuts in \mathcal{A} , we obtain that f_{11} is continuous in 0. Since $\mathcal{A}^{\mathcal{R}}$ is a $*$ -algebra, $f \in \mathcal{A}^{\mathcal{R}}$ implies $|f|^2 \in \mathcal{A}^{\mathcal{R}}$, hence $|f_{11}|^2 + |f_{12}|^2$ is continuous in 0. Since f_{12} vanishes in 0, f_{12} (and analogously f_{21}) is continuous in 0. On the other hand, let g be a real valued Riemann measurable function on $[0, 1]$ w.r.t. Lebesgue measure s.t. $g(0) = 0$ and $|g| \leq 1$ and consider the matrix $f := \begin{pmatrix} 0 & g \\ 0 & 0 \end{pmatrix}$. For any $\varepsilon > 0$ set

$$f_\varepsilon^\pm(t) := \begin{cases} \begin{pmatrix} 0 & g(t) \\ g(t) & 0 \end{pmatrix}, & t > \varepsilon \\ \begin{pmatrix} \pm \varepsilon^2 & 0 \\ 0 & \pm \varepsilon^{-2} \end{pmatrix}, & t \leq \varepsilon. \end{cases}$$

We easily get $f_\varepsilon^\pm \in \mathcal{A}^{\mathcal{R}}$, $f_\varepsilon^- \leq f \leq f_\varepsilon^+$ and $\tau(f_\varepsilon^+ - f_\varepsilon^-) = 2\varepsilon + \varepsilon^3$, namely f is separating for \mathcal{U} -cuts in $\mathcal{A}^{\mathcal{R}}$, hence $f \in \mathcal{A}^{\mathcal{U}}$. This shows at once that $\mathcal{A}^{\mathcal{R}}$ is strictly smaller than $\mathcal{A}^{\mathcal{U}}$ and that the latter is not an algebra, since, if g is not continuous in 0, $(f^2)_{11}$ is not continuous in 0 hence does not belong to $\mathcal{A}^{\mathcal{U}}$. The rest of the proof follows with analogous arguments. ■

2.2. Densely Defined Trace on a Non-unital C*-Algebra

The construction considered in this subsection corresponds to the most general case described in the abelian setting, namely the case of a Radon measure on a non-compact space. There the local structure, namely the structure given by the compact support functions, plays a crucial role, and the same will be in our construction.

In the classical case, a function is said Riemann integrable if it has compact (or τ -finite) support, and is continuous but for a null set. Given a C*-algebra with a semicontinuous densely defined trace, we define a suitable analogue of this family, and show that its norm closure coincides with the \mathcal{R} -closure of the given C*-algebra. Therefore on such elements, the integral may be defined in two equivalent ways. Either as a separating element, as explained before, or as a limit (when it exists) of the τ -finite restrictions (see below).

In the nonabelian setting, the local structure is determined by a suitable family of projections which has to be closed w.r.t. the “ \vee ” operation. We present here a family of projections with these properties; we shall show at

the end of this subsection that any other reasonable family would have produced the same result.

In this subsection \mathcal{A} denotes a non-unital C^* -algebra acting on a Hilbert space \mathcal{H} , equipped with a semicontinuous densely defined trace τ .

LEMMA 2.18. *Any projection in $\mathcal{A}^{\mathcal{R}}$ has finite trace.*

Proof. Let e be a projection in $\mathcal{A}^{\mathcal{R}}$, $a \in \mathcal{A}$ such that $e \leq a$, and $a_n \in \mathcal{A}_+$ with finite trace with $a_n \rightarrow a$ in norm. Then $ea_n e \rightarrow eae$, hence there exists an n such that $ea_n e \geq eae - 1/2e \geq 1/2e$, from which the thesis follows. ■

DEFINITION 2.19. Let us denote by \mathcal{E} the minimal family of projections in \mathcal{A}'' containing all the Riemann measurable spectral projections of the selfadjoint elements in \mathcal{A} , and closed w.r.t. the “ \vee ” operation.

LEMMA 2.20. *The family \mathcal{E} is contained in $\mathcal{A}^{\mathcal{R}}$.*

Proof. Let us consider the family \mathcal{E}_1 of projections $e \in \mathcal{A}^{\mathcal{R}}$ which are separating for \mathcal{R} -cuts $\{a_\varepsilon^\pm\}_{\varepsilon>0}$ such that $a_\varepsilon^- \geq 0$ and $\tau(\text{supp}(a_\varepsilon^+) - e) < \varepsilon$. The τ -finite Riemann measurable spectral projections of a selfadjoint element $x \in \mathcal{A}$ belong to \mathcal{E}_1 by Lemma 6.4 applied to $(\sigma(x), \mu_x)$.

Now we show that \mathcal{E}_1 is closed w.r.t. the \vee operation. Let e, f be in \mathcal{E}_1 , with $\{a_\varepsilon^\pm\}_{\varepsilon>0}$, $\{b_\varepsilon^\pm\}_{\varepsilon>0}$ the corresponding cuts. Then $\tau(e - \text{supp}(a_\varepsilon^-)) \leq \varepsilon$ and the same for f , hence $\tau(e \vee f - \text{supp}(a_\varepsilon^-) \vee \text{supp}(b_\varepsilon^-)) \leq 2\varepsilon$. Since for any positive element y with τ -finite support we have $\tau(\text{supp}(y) - y^{1/n}) \rightarrow 0$ and if y_1, y_2 are positive elements then $\text{supp}(y_1) \vee \text{supp}(y_2) = \text{supp}(y_1 + y_2)$, we get an n such that $\tau(e \vee f - ((a_\varepsilon^- + b_\varepsilon^-)/2)^{1/n}) \leq 3\varepsilon$. As for the approximation from above, we observe that $\tau(\text{supp}(a_\varepsilon^+) \vee \text{supp}(b_\varepsilon^+) - e \vee f) \leq 2\varepsilon$, and there exists an n such that $\tau((a_\varepsilon^+ + b_\varepsilon^+)^{1/n} - \text{supp}(a_\varepsilon^+ + b_\varepsilon^+)) \leq \varepsilon$, hence $\tau((a_\varepsilon^+ + b_\varepsilon^+)^{1/n} - e \vee f) \leq 3\varepsilon$. This shows that $e \vee f \in \mathcal{E}_1$. Since \mathcal{E}_1 is contained in $\mathcal{A}^{\mathcal{R}}$, the thesis follows by the minimality of \mathcal{E} . ■

Let e be a projection in $\mathcal{A}^{\mathcal{R}}$, and consider the C^* -algebra $\mathcal{A}_e := \{a \in \mathcal{A} : ae = ea = a\}$. Since $\mathcal{A}^{\mathcal{R}}$ is a bimodule and contains e , we have $\mathcal{A}_e + \mathbb{C}e \subset \mathcal{A}^{\mathcal{R}}$, hence $(\mathcal{A}_e + \mathbb{C}e)^{\mathcal{R}} \subset \mathcal{A}''$. By Lemma 2.18, projections in $\mathcal{A}^{\mathcal{R}}$ are τ -finite. Therefore, by the results of the previous subsection, $(\mathcal{A}_e + \mathbb{C}e)^{\mathcal{R}}$ is an \mathcal{R} -closed C^* -algebra in \mathcal{A}'' . By Lemma 2.10 we conclude that $(\mathcal{A}_e + \mathbb{C}e)^{\mathcal{R}}$ is indeed contained in $\mathcal{A}^{\mathcal{R}}$. We may therefore consider the minimal \mathcal{R} -closed C^* -algebra \mathcal{R}_e verifying $\mathcal{A}_e \subseteq \mathcal{R}_e \subset \mathcal{A}^{\mathcal{R}}$. By definition, the map $e \rightarrow \mathcal{R}_e$ is order preserving, therefore the union $\bigcup_{e \in \mathcal{E}} \mathcal{R}_e$ is a $*$ -algebra, since (\mathcal{E}, \vee) is a directed set. We shall denote by $\mathcal{R}_{\mathcal{E}}$ the C^* -algebra given by the norm closure of $\bigcup_{e \in \mathcal{E}} \mathcal{R}_e$. Then the following holds.

THEOREM 2.21. *Let (\mathcal{A}, τ) be a C^* -algebra with a semicontinuous densely defined trace, acting on a Hilbert space \mathcal{H} . Then $\mathcal{R}_{\mathcal{E}} = \mathcal{A}^{\mathcal{R}}$.*

First we observe that such result immediately implies Theorem 2.4 in the densely defined case, by Lemma 2.13. Before proving the Theorem, we need a simple Lemma.

LEMMA 2.22. *Let e be a projection on \mathcal{H} , and x a selfadjoint operator s.t. $0 \leq x \leq \alpha e + \beta e^{\perp}$, $\alpha, \beta \geq 0$. Then $\|e^{\perp} x e\| \leq \sqrt{\alpha \beta}$*

Proof. Let η_1 be a unit vector in $e\mathcal{H}$, η_2 a unit vector in $e^{\perp}\mathcal{H}$, $\vartheta \in (0, \pi/2)$. then

$$\begin{aligned} & 2 \sin \vartheta \cos \vartheta \operatorname{Re}(\eta_1, x \eta_2) \\ &= ((\cos \vartheta \eta_1 + \sin \vartheta \eta_2), (e^{\perp} x e + e x e^{\perp})) (\cos \vartheta \eta_1 + \sin \vartheta \eta_2)) \\ &\leq (\alpha \cos^2 \vartheta + \beta \sin^2 \vartheta) \end{aligned}$$

which gives $\operatorname{Re}(\eta_1, x \eta_2) \leq \alpha \cot \vartheta + \beta \tan \vartheta$. Taking the supremum of the left hand side w.r.t. η_1, η_2 and the infimum of the right hand side w.r.t. $\vartheta \in (0, \pi/2)$ we immediately get the thesis. ■

Proof of Theorem 2.21 ($\mathcal{R}_{\mathcal{E}} \subset \mathcal{A}^{\mathcal{R}}$). By the results of the preceding subsection, it follows that $\bigcup_{e \in \mathcal{E}} \mathcal{R}_e$ is a vector space in $\mathcal{A}^{\mathcal{T}}$, hence its norm closure $\mathcal{R}_{\mathcal{E}}$ is contained in $\mathcal{A}^{\mathcal{R}}$ by Lemma 2.11.

($\mathcal{A}^{\mathcal{R}} \subset \mathcal{R}_{\mathcal{E}}$.) Let $x \in \mathcal{A}_{\text{sa}}^{\mathcal{R}}$, and $\{a_e^{\pm}\}_{e \in \mathcal{E}}$ the corresponding \mathcal{R} -cut in \mathcal{A} . We observe that it is not restrictive to assume $x \geq 0$, possibly replacing x with $x - a_1^-$. Then we set $x_{\delta} := \phi_{\delta} x \phi_{\delta}$, where $\phi_{\delta} := \phi_{\delta}(a_1^+)$ denotes the mollified spectral projection defined in Lemma 2.2, and observe that $\{\phi_{\delta} a_e^{\pm} \phi_{\delta}\}_{e \in \mathcal{E}}$ gives an \mathcal{R} -cut for x_{δ} . Since $\phi_{\delta} a_e^{\pm} \phi_{\delta} \in \mathcal{A}_{e_{\delta}}$, with $e_{\delta} = \chi_{[\delta/2, \infty)}(a_1^+)$, we have $x_{\delta} \in \bigcup_{e \in \mathcal{E}} \mathcal{R}_e$, for any $\delta > 0$. The theorem is proved if we show that $x_{\delta} \rightarrow x$ when $\delta \rightarrow 0$. Indeed,

$$\begin{aligned} \|x - x_{\delta}\| &\leq \|(1 - \phi_{\delta})(1 - \phi_{\delta})\| + 2 \|(1 - \phi_{\delta})x\| \leq 3 \|e_{\delta}^{\perp} x\| \\ &\leq 3 \|e_{\delta}^{\perp} x e_{\delta}^{\perp}\| + 3 \|e_{\delta}^{\perp} x e_{\delta}\| \leq 3 \frac{\delta}{2} + 6 \sqrt{\frac{\delta}{2}} \sqrt{\|a_1^+\|}, \end{aligned}$$

where the last inequality relies on the estimates $\|e_{\delta}^{\perp} x e_{\delta}^{\perp}\| \leq \|e_{\delta}^{\perp} a_1^+ e_{\delta}^{\perp}\| \leq \frac{\delta}{2}$ and $x \leq a_1^+ \leq \frac{\delta}{2} e_{\delta}^{\perp} + \|a_1^+\| e_{\delta}$, together with Lemma 2.22. ■

We say that $a \in \mathcal{A}^{\mathcal{R}}$ is \mathcal{R} -summable if at least one of a^+ or a^- is τ -finite. Then we may prove.

PROPOSITION 2.23. *a is \mathcal{R} -summable iff there exists $\lim_{e \in \mathcal{A}^{\mathcal{R}}} \tau(eae)$. When a is positive or τ -finite, the above limit coincides with $\tau(a)$.*

As explained before, τ coincides with the pull-back via ρ_π of the trace on the GNS representation, hence it is a semicontinuous semifinite trace on $\mathcal{A}^{\mathcal{R}}$.

PROPOSITION 2.24. *If τ is densely defined on \mathcal{A} , then $(\mathcal{A}^{\mathcal{R}}, \tau)$ is a Riemann algebra with a semicontinuous, densely defined trace.*

Proof. τ is densely defined on each \mathcal{R}_e , hence the thesis follows. ▀

Remark 2.25. Previous Theorem shows that the algebra $\mathcal{R}_{\mathcal{E}}$ may be defined independently of the family \mathcal{E} . We note that, replacing $\mathcal{E} := \mathcal{E}_0$ with other reasonable families, the same algebra is obtained. Indeed we may consider the set \mathcal{E}_1 considered in the proof of Lemma 2.20, the set \mathcal{E}_2 of all projections in $\mathcal{A}^{\mathcal{R}}$, the set \mathcal{E}_3 of all compact support projections in \mathcal{A}'' , namely projections which are majorized by an element in \mathcal{A} (cf. [18]), or the set \mathcal{E}_4 of all τ -finite projections in \mathcal{A}'' . It is easy to see that $\mathcal{E}_0 \subseteq \mathcal{E}_1 \subseteq \mathcal{E}_2 \subseteq \mathcal{E}_3 \subseteq \mathcal{E}_4$, and all such sets are directed. If e is a τ -finite projection, \mathcal{R}_e can be defined as the minimal Riemann algebra containing \mathcal{A}_e , the set of such algebras being non empty, containing at least $\mathcal{A}^{\mathcal{R}}$. Then for any family \mathcal{E}_i , we may define the C^* -algebra $\mathcal{R}_i := \overline{\bigcup_{e \in \mathcal{E}_i} \mathcal{R}_e}$, and obtain $\mathcal{A}^{\mathcal{R}} = \mathcal{R}_0 \subseteq \mathcal{R}_1 \subseteq \mathcal{R}_2 \subseteq \mathcal{R}_3 \subseteq \mathcal{R}_4$. Since on the other hand \mathcal{R}_4 is contained in $\mathcal{A}^{\mathcal{R}}$, we have proved that all these algebras coincide.

Let e be a τ -finite projection in \mathcal{A}'' . We say that an \mathcal{R} -cut is e -supported if all elements in A^- and A^+ belong to \mathcal{A}_e . Then we proved that, for any $i = 0, 1, 2, 3, 4$, $\mathcal{A}^{\mathcal{R}}$ is the norm closure of the Dedekind completion of \mathcal{A} w.r.t. \mathcal{E}_i -supported \mathcal{R} -cuts.

2.3. Semifinite Non-densely Defined Trace

In the rest of the section we consider semifinite non-densely defined traces, namely pairs (\mathcal{A}, τ) for which the closure \mathcal{A}_0 of the domain of the trace is a proper ideal in \mathcal{A} .

PROPOSITION 2.26. *Then \mathcal{R} -closure $\mathcal{A}_0^{\mathcal{R}}$ of \mathcal{A}_0 is a bimodule on \mathcal{A} .*

Proof. Let $x \in (\mathcal{A}_0^{\mathcal{R}})_{\text{sa}}$, then there is an \mathcal{R} -cut $\{a_e^\pm\}$ in \mathcal{A}_0 for which x is a separating element. Let $a \in \mathcal{A}$, then a^*xa is a separating element for the \mathcal{R} -cut $\{a^*a_e^\pm a\}$ in \mathcal{A}_0 , so that $a^*xa \in (\mathcal{A}_0^{\mathcal{R}})_{\text{sa}}$. Now we have to show that if $x \in (\mathcal{A}_0^{\mathcal{R}})_{\text{sa}}$, $y \in \mathcal{A}_{\text{sa}}^{\mathcal{R}}$, then $xy + yx$ and $i(xy - yx) \in (\mathcal{A}_0^{\mathcal{R}})_{\text{sa}}$. Because of Eqs. (2.3) and (2.4) the thesis follows by the previous result.

THEOREM 2.27. *The set $\mathcal{A}_0^{\mathcal{R}} + \mathcal{A}$ is a C^* -algebra, therefore coincides with $\mathcal{A}^{\mathcal{R}}$ and Theorem 2.4 follows for \mathcal{A} .*

Proof. Since $\mathcal{A}_0^{\mathcal{R}}$ is a $*$ -bimodule on \mathcal{A} , $\mathcal{A}_0^{\mathcal{R}} + \mathcal{A}$ is a $*$ -algebra. Moreover, $\mathcal{A}_0^{\mathcal{R}}$ is a closed ideal in the closure of $\mathcal{A}_0^{\mathcal{R}} + \mathcal{A}$, hence, by [18, Corollary 1.5.8], $\mathcal{A}_0^{\mathcal{R}} + \mathcal{A}$ is a C^* -algebra. Finally we show that $\mathcal{A}^{\mathcal{R}} \subset \mathcal{A}_0^{\mathcal{R}} + \mathcal{A}$, and the thesis will follow from Lemma 2.13. Indeed, if $x \in \mathcal{A}_{\text{sa}}^{\mathcal{R}}$, it is a separating element for an \mathcal{R} -cut $\{a_\varepsilon^\pm\}$ in \mathcal{A}_{sa} . Then $x - a_1^-$ is a separating element for the \mathcal{R} -cut $\{a_\varepsilon^\pm - a_1^-\}$ in $(\mathcal{A}_0)_{\text{sa}}$ because $-\varepsilon \leq \tau(a_\varepsilon^- - a_1^-) < 1$, and $0 \leq \tau(a_\varepsilon^+ - a_1^-) < 1 + \varepsilon$. ■

LEMMA 2.28. *If $x \in (\mathcal{A}_0^{\mathcal{R}} + \mathcal{A})_+$ and $\tau(x)$ is finite, then $x \in \mathcal{A}_0^{\mathcal{R}}$.*

Proof. Let $x = r + a$, $r \in (\mathcal{A}_0^{\mathcal{R}})_{\text{sa}}$, $a \in \mathcal{A}_{\text{sa}}$. Since τ is densely defined on $\mathcal{A}_0^{\mathcal{R}}$ by Theorem 2.24, $r \in \overline{\mathcal{J}(\tau|_{\mathcal{A}_0^{\mathcal{R}}})} \subset \overline{\mathcal{J}(\tau|_{\mathcal{A}_0^{\mathcal{R}} + \mathcal{A}})}$, hence $a \in \overline{\mathcal{J}(\tau|_{\mathcal{A}_0^{\mathcal{R}} + \mathcal{A}})}$. By Lemma 2.2 we get $\tau(\phi_\varepsilon(|a|)) < \infty$ for any $\varepsilon > 0$, hence $a \in \overline{\mathcal{J}(\tau|_{\mathcal{A}})} = \mathcal{A}_0$, and the thesis follows. ■

COROLLARY 2.29. *The \mathcal{R} -closure of a C^* -algebra \mathcal{A} with a semicontinuous semifinite trace coincides with $\mathcal{A}_0^{\mathcal{R}} + \mathcal{A}$. The closure of the domain of $\tau|_{\mathcal{A}^{\mathcal{R}}}$ coincides with $\mathcal{A}_0^{\mathcal{R}}$.*

Proof. The first statement follows by Theorem 2.27. The second statement is an immediate consequence of Lemma 2.28. ■

3. UNBOUNDED RIEMANN INTEGRATION

In this Section we give the construction of the unbounded Riemann measurable elements. They form a family which is closed under the $\mathcal{A}^{\mathcal{R}} - \mathcal{A}^{\mathcal{R}}$ $*$ -bimodule operations, even if the $*$ -bimodule properties only hold τ -a.e. The product of two unbounded Riemann measurable elements is not Riemann measurable in general, but, if S, T are Riemann measurable, there is \tilde{T} , τ -a.e. equivalent to T , s.t. $S\tilde{T}$ is Riemann measurable.

As a matter of fact, this construction will be performed on a general Riemann algebra with a semicontinuous semifinite trace, and then particularized to the \mathcal{R} -closure of a C^* -algebra. We begin with technical results on unbounded operators and operations on them, needed in the sequel.

Let us denote by $\mathcal{L}(\mathcal{H})$ the set of linear operators on \mathcal{H} , neither necessarily bounded nor closed, with $\mathcal{C}(\mathcal{H}) \subset \mathcal{L}(\mathcal{H})$ the set of closed, densely defined operators, and with $\mathcal{B}(\mathcal{H})$, as usual, the C^* -algebra of bounded operators. Let $T \in \mathcal{L}(\mathcal{H})$, then T is densely defined as an

operator from $\mathcal{H} := \overline{\mathcal{D}(T)}$ to \mathcal{H} , and we can take its adjoint $T^*: \mathcal{D}(T^*) \subset \mathcal{H} \rightarrow \mathcal{H}$. Set

$$T^+ \xi := \begin{cases} T^* \xi, & \xi \in \mathcal{D}(T^*) \\ 0, & \xi \in \mathcal{D}(T^*)^\perp, \end{cases}$$

extended by linearity.

LEMMA 3.1. $T^+ \in \mathcal{C}(\mathcal{H})$.

Proof. Let $\{\xi_n\} \subset \mathcal{D}(T^+)$, $\xi_n \rightarrow \xi$, $T^+ \xi_n \rightarrow \eta$, and let p be the projection onto $\overline{\mathcal{D}(T^*)}$. Then $p\xi_n \in \mathcal{D}(T^*)$, $p\xi_n \rightarrow p\xi$ and $p^\perp \xi_n \rightarrow p^\perp \xi \in \mathcal{D}(T^*)^\perp$, so that $T^* p\xi_n = T^+ p\xi_n + T^+ p^\perp \xi_n = T^+ \xi_n \rightarrow \eta$. As T^* is closed, we get $p\xi \in \mathcal{D}(T^*)$ and $T^* p\xi = \eta$, so that $\xi = p\xi + p^\perp \xi \in \mathcal{D}(T^+)$ and $T^+ \xi = T^+ p\xi = T^* p\xi = \eta$. ■

DEFINITION 3.2. For any linear operator T its *natural extension* is the closed operator $T^\natural := (T^+)^*$. Introduce the set of *locally bounded* operators, w.r.t. a projection e , $\mathcal{L}_0(e) := \{T \in \mathcal{L}(\mathcal{H}) : \text{there is an increasing sequence of projections } e_n \nearrow e \text{ s.t. } \mathcal{H}_0 := \bigcup e_n \mathcal{H} \subset \mathcal{D}(T) \cap \mathcal{D}(T^*), eTe_n, e_n Te \in \mathcal{B}(\mathcal{H})\}$.

We want to show that the natural extension of a locally bounded linear operator is locally bounded as well. For the elementary rules of calculus with unbounded operators we refer the reader to [19] (in particular [19, Chap. 8]).

PROPOSITION 3.3. *Let T be a locally bounded operator w.r.t. e . Then T^+ and T^\natural are locally bounded w.r.t. e , too. Moreover, setting $T_m := \overline{(eTe)|_{\mathcal{H}_0}}$ and $T_M := ((eTe)^*|_{\mathcal{H}_0})^*$, T_m , T_M are closed operators s.t. $(T_M)^* = (T^*)_m$, and $T_m \subset \overline{eT^\natural e} \subset T_M$.*

Proof. We divide it in steps.

(i) $\mathcal{D}(Te)$ is dense in \mathcal{H} . As $\mathcal{D}(Te) = \{\xi \in \mathcal{H} : e\xi \in \mathcal{D}(T)\} \equiv \mathcal{D}(T) \cap e\mathcal{H} \oplus e^\perp \mathcal{H}$ and $\mathcal{D}(T) \cap e\mathcal{H} \supset \mathcal{H}_0$ is dense in $e\mathcal{H}$, we get the thesis.

(ii) eTe is closable and densely defined. Indeed, eTe is densely defined because of (i). Let $\{\xi_n\} \subset \mathcal{D}(eTe)$, $\xi_n \rightarrow 0$, $eTe\xi_n \rightarrow \eta$. Then $e_m eTe\xi_n = e_m Te\xi_n \rightarrow 0$, because $e_m Te$ is bounded. On the other hand $e_m eTe\xi_n \rightarrow e_m \eta$, so that $e_m \eta = 0$ for all $m \in \mathbb{N}$. Therefore $\eta = e\eta = 0$.

(iii) $(eTe)^*|_{\mathcal{H}_0} = (eT^+ e)|_{\mathcal{H}_0}$. Let us set $\mathcal{K} := \overline{\mathcal{D}(T)}$ and observe that $eT^* \equiv e^* T^* : \mathcal{D}(T^*) \subset \mathcal{H} \rightarrow \mathcal{H} \hookrightarrow \mathcal{K}$ is extended by $(Te)^*$, so that $(e_n Te)^* = (Te)^* e_n \supset eT^* e_n$, where the equality holds because Te is densely defined. Therefore $eT^+ e_n \subset (e_n Te)^*$, so that equality holds because $eT^+ e_n$

is bounded and everywhere defined. Finally $(eTe)^*e_n = (Te)^*ee_n = (Te)^*e_n = eT^+e_n$ which implies the thesis.

(iv) $e_nT^+e \subset (eTe_n)^*$. Let $\xi \in \mathcal{D}(T^*) \cap e\mathcal{H}$, $\eta \in \mathcal{H}$, then

$$\begin{aligned} (\eta, e_nT^+e\xi) &= (\eta, e_nT^+\xi) = (\eta, e_nT^*\xi) = (e_n\eta, T^*\xi) \\ &= (Te_n\eta, \xi) = (Te_n\eta, e\xi) = (eTe_n\eta, \xi) = (\eta, (eTe_n)^*\xi) \end{aligned}$$

because $e_n\eta \in \mathcal{D}(T)$ and eTe_n is bounded.

(v) $eTe|_{\mathcal{H}_0} = eT^{\natural}e|_{\mathcal{H}_0}$. As $e_nT^+e \subset (eTe_n)^* \Rightarrow eTe_n \subset (e_nT^+e)^*$, we get $eTe_n = (e_nT^+e)^* \supset e(e_nT^+)^* = eT^{\natural}e_n$. But $e_n\mathcal{H} \subset \mathcal{D}(T^{\natural})$, implies $eT^{\natural}e_n \in \mathcal{B}(\mathcal{H})$, and the thesis follows.

Therefore, the first statement of the proposition follows from $eT^+e_n = (e_nTe)^*$ and $e_nT^+e \subset (eTe_n)^*$. From (ii) it follows that eT^+e is closable, therefore $\overline{eT^+e} \supset \overline{(eT^+e)|_{\mathcal{H}_0}} = \overline{(eT^*e)|_{\mathcal{H}_0}} = T_M^*$, so that $T_M \supset (eT^+e)^* \supset e(eT^+)^* = eT^{\natural}e$. Finally $eT^{\natural}e \supset eT^{\natural}e|_{\mathcal{H}_0} = \overline{eTe|_{\mathcal{H}_0}} = T_m$. ■

Let $\mathcal{R} \subset \mathcal{B}(\mathcal{H})$ be a Riemann algebra w.r.t. a semicontinuous semifinite trace τ . Inspired by Christensen [3] we now introduce the set of essentially τ -measurable operators (a subset of the locally bounded ones). Recall that $T \hat{\in} \mathcal{R}''$ stands for $u'T \subset Tu'$ for any unitary operator $u' \in \mathcal{R}'$.

DEFINITION 3.4. A sequence $\{e_n\}$ of projections in $\mathcal{B}(\mathcal{H})$ is called a *strongly dense domain* (SDD) w.r.t. (\mathcal{R}, τ) , if $e_n^\perp \in \mathcal{R}$ is τ -finite, $\tau(e_n^\perp) \rightarrow 0$. Then $e^\perp := \inf_n e_n^\perp \in \mathcal{R}$, because of \mathcal{R} -closedness. If $T \hat{\in} \mathcal{R}''$, we say $\{e_n\}$ is a SDD for T if $\mathcal{H}_0 := \bigcup e_n\mathcal{H} \subset \mathcal{D}(T) \cap \mathcal{D}(T^*)$ and $eTe_n, e_nTe \in \mathcal{R}$. Let us introduce the set of *essentially τ -measurable* operators $\bar{\mathcal{R}}_0 := \{T \hat{\in} \mathcal{R}'' : \text{there is a SDD for } T\}$, and the set of τ -measurable operators $\bar{\mathcal{R}} := \bar{\mathcal{R}}_0 \cap \mathcal{C}(\mathcal{H})$. Let $S, T \hat{\in} \mathcal{R}''$, we say that $S = T$ almost everywhere if there is a common SDD $\{e_n\}$ for S and T s.t. $eSe|_{\mathcal{H}_0} = eTe|_{\mathcal{H}_0}$.

Remark 3.5. In order for $\bar{\mathcal{R}}$ to be larger than \mathcal{R} , one needs SDDs for which $\tau(e_n^\perp) > 0$ for any n . This is not always the case. For example the compact operators with the usual trace form a Riemann algebra for which $\bar{\mathcal{R}} = \mathcal{R}$.

LEMMA 3.6. Let \mathcal{R} be a C^* -algebra, e, f projections in \mathcal{R} . Then $e \vee f, e \wedge f \in \mathcal{R}$.

Proof. As $e \vee f \leq e + f \leq 2(e \vee f)$ and $t \rightarrow t^{1/n}$ is operator-increasing, it follows $e \vee f \leq (e + f)^{1/n} \leq 2^{1/n}(e \vee f)$. Therefore $(e + f)^{1/n} \rightarrow e \vee f$ and $e \vee f \in \mathcal{R}$. Besides, in the unitalization of \mathcal{R} , $e \wedge f = 1 - (1 - e) \vee (1 - f) = \lim_n 1 - (2 - e - f)^{1/n} = \lim_n 1 - 2^{1/n} \sum_{k=0}^{\infty} (-1)^k \binom{1/n}{k} (\frac{e+f}{2})^{1/n} = \lim_n e_n$, where $e_n := 2^{1/n} \sum_{k=1}^{\infty} (-1)^{k-1} \binom{1/n}{k} (\frac{e+f}{2})^{1/n} \in \mathcal{R}$. ■

LEMMA 3.7. *Let $T \in \bar{\mathcal{R}}_0$, $\{e_n\}$, $\{f_n\}$ SDD for T . Then $g_n := e_n \wedge f_n$ is an SDD for T .*

Proof. Because of Lemma 3.6, $g_n^\perp \in \mathcal{R}$, and $\tau(g_n^\perp) \rightarrow 0$, that is $\{g_n\}$ is an SDD. Besides $g_n T g = g_n e_n T e g = e_n T e g - g_n^\perp e_n T e g = e_n T e - e_n T e g^\perp - g_n^\perp e_n T e + g_n^\perp e_n T e g^\perp \in \mathcal{R}$, and the thesis follows. ■

PROPOSITIONS 3.8. (i) $T \in \mathcal{L}(\mathcal{H})$, $T \hat{\in} \mathcal{R}'' \Rightarrow T^+ , T^\natural \hat{\in} \mathcal{R}''$.

(ii) *Equality almost everywhere is an equivalence relation.*

(iii) $T \in \bar{\mathcal{R}}_0 \Rightarrow T = T^\natural$ almost everywhere and $T^\natural \in \bar{\mathcal{R}}_0$.

Proof. (i) Let $u' \in \mathcal{R}'$ be unitary, then $u' T \subset T u'$ so that $u' T^* \subset T^* u'$, therefore $u' \mathcal{D}(T^*) = \mathcal{D}(T^*)$. Moreover for $\eta \in \mathcal{D}(T^*)^\perp = u' \mathcal{D}(T^*)^\perp$ we get $T^+ u' \eta = 0 = u' T^+ \eta$. In all $T^+ \hat{\in} \mathcal{R}''$, so that $T^\natural \hat{\in} \mathcal{R}''$ as well.

(ii) This is obvious.

(iii) As a SDD for T is also a SDD for T^\natural , the thesis follows from (i) and (v) in the proof of Proposition 3.3. ■

In particular we have just proved that the equivalence class of an operator in $\bar{\mathcal{R}}_0$ contains a closed operator, hence $\bar{\mathcal{R}}_0 / \sim = \bar{\mathcal{R}} / \sim$.

DEFINITION 3.9. For $S, T \in \bar{\mathcal{R}}$, $a \in \mathcal{R}$ define $S \oplus T := (S + T)^\natural$, $a \odot T := (aT)^\natural$, $T \odot a := (Ta)^\natural$.

THEOREM 3.10. $\bar{\mathcal{R}}$ is an almost everywhere $*$ -bimodule over \mathcal{R} , w.r.t. strong sense operations, namely $S, T \in \bar{\mathcal{R}}$, $a \in \mathcal{R} \Rightarrow S \oplus T \in \bar{\mathcal{R}}$, $a \odot T, T \odot a \in \bar{\mathcal{R}}$, and the bimodule properties hold almost everywhere.

Proof. Let us first prove

(i) $S, T \in \bar{\mathcal{R}}_0 \Rightarrow S + T \in \bar{\mathcal{R}}_0$ and $(S + T)^\natural = S^\natural + T^\natural$ almost everywhere. Assume $\{e_n\}$, $\{f_n\}$ are SDD for S and T respectively, and set $g_n := e_n \wedge f_n$. Then $\{g_n\}$ is a common SDD for S and T , as in Lemma 3.7. Besides $g_n(S + T)g = g_n e_n S e g + g_n f_n T f g \in \mathcal{R}$, $g(S + T)g_n = g e S e_n g_n + g f T f_n g_n \in \mathcal{R}$, as in Lemma 3.7. Finally $g(S + T)^\natural g|_{\mathcal{H}_0} = g(S + T)g|_{\mathcal{H}_0} = g S g|_{\mathcal{H}_0} + g T g|_{\mathcal{H}_0} = g(S^\natural + T^\natural)g|_{\mathcal{H}_0}$.

Then we prove

(ii) $T \in \bar{\mathcal{R}}_0$, $a \in \mathcal{R} \Rightarrow aT, Ta \in \bar{\mathcal{R}}_0$, and $(aT)^\natural = aT^\natural$, $(Ta)^\natural = T^\natural a$ almost everywhere.

Using (i), we need only prove it for $a = u$ a unitary operator. Set $g_n := e_n \wedge u^* e_n u \wedge u e_n u^*$. Then g_n is a common SDD for T, uT, Tu , as in Lemma 3.7, and $g_n T u g = g_n e_n T u u^* e u g \in \mathcal{R}$, $g T u g_n = g e T u u^* e_n u g_n \in \mathcal{R}$, as

in Lemma 3.7. Analogously $g_n u T g \in \mathcal{R}$, $g u T g_n \in \mathcal{R}$. The last two statements are proved as in (i).

Now the theorem follows by (i), (ii), and Proposition 3.8. ■

Observe that the previous theorem generalises results by Segal [24]; indeed, if \mathcal{R} is a von Neumann algebra and τ a normal semifinite faithful trace on it, equality almost everywhere turns out to be equality (cf. [24, Corollary 5.1]) and the two notions of strong sense operations coincide.

PROPOSITION 3.11. *If $S, T \in \bar{\mathcal{R}}$ and $\{e_n\}$ is an SDD both for S and T then SeT and TeS belong to $\bar{\mathcal{R}}$. In particular, there exists \tilde{T} τ -a.e. equivalent to T such that $S\tilde{T}$ and $\tilde{T}S$ belong to $\bar{\mathcal{R}}$.*

Proof. Let $eTe_n = \sum_i \lambda_{ni} u_{ni}$, $e_n Se = \sum_i \mu_{ni} v_{ni}$ be decompositions of eTe_n and $e_n Se$ into linear combinations of unitaries in \mathcal{R} , and set $f_n := \wedge_i u_{ni}^* e_n u_{ni} \wedge e_n \wedge_i v_{ni} e_n v_{ni}^*$. Then $f_n \mathcal{H} \subset \mathcal{D}(T)$ and

$$\begin{aligned} eTf_n &= eTe_n f_n = \sum_i \lambda_{ni} u_{ni} f_n = \sum_i \lambda_{ni} e_n u_{ni} f_n \\ &= e_n (eTe_n) f_n, \end{aligned}$$

as a consequence $eTf_n \mathcal{H} \subset \mathcal{D}(S)$, then $fSeTf_n = f(eSe_n)(eTe_n) f_n \in \mathcal{R}$. Repeating this argument for $f_n SeTf$ we show that $\{f_n\}$ is an SDD for SeT . Since $\tilde{T} := \overline{eTe}$ is τ -a.e. equivalent to T the last statement follows. ■

From now on we denote by ρ the GNS representation of τ , and with $\mathcal{M} := \rho(\mathcal{R})''$. We want to extend ρ to a morphism of $\bar{\mathcal{R}}$ to the $*$ -algebra $\bar{\mathcal{M}}$ of τ -measurable operators affiliated with \mathcal{M} [24]. Let us recall that the topology of convergence in measure in $\bar{\mathcal{M}}$ is generated by the neighborhood basis $\{V(\varepsilon, \delta)\}_{\varepsilon, \delta > 0}$, where $V(\varepsilon, \delta) := \{T \in \bar{\mathcal{M}} : \text{there is a projection } p \in \mathcal{M} \text{ s.t. } \tau(p^\perp) < \delta, \|Tp\| < \varepsilon\}$. Let us set $T = \tau \lim T_n$ for $T_n \rightarrow T$ in measure.

PROPOSITION 3.12. *Let $T \in \bar{\mathcal{R}}$.*

- (i) *If $\{e_n\}$ is an SDD for T , then $\tau \lim \rho(eTe_n)$ and $\tau \lim \rho(e_n Te)$ exist and are equal.*
- (ii) *$\rho(T) := \tau \lim \rho(eTe_n)$ does not depend on the SDD $\{e_n\}$ and belongs to $\bar{\mathcal{M}}$.*

Proof. (i) The sequence $\{\rho(eTe_n)\}$ is easily seen to be Cauchy in measure. Observe that from [25, Theorem 3.7], $\tau \lim \rho(e_n Te_n) = \tau \lim (1 - \rho(e_n^\perp)) \rho(eTe_n) = \tau \lim \rho(eTe_n)$. Finally $\tau \lim \rho(e_n Te) = \tau \lim \rho(e_n Te_n)$,

because for all $\delta > 0$, let $k \in \mathbb{N}$ be s.t. $\tau(e_k^\perp) < \delta$, and $n > k$, then $\|[\rho(e_n Te) - \rho(e_n Te_n)](1 - \rho(e_k^\perp))\| = \|\rho(e_n Te_k) - \rho(e_n Te_n)\| = 0$. The thesis follows.

(ii) Let $\{f_n\}$ be another SDD for T and set $g_n := e_n \wedge f_n$. Then $\tau(g_n^\perp) \rightarrow 0$, and $\|(\rho(eTe_n) - \rho(fTf_n))(1 - \rho(g_n^\perp))\| \leq \|\rho(e - g)\rho(eTe_n)(1 - \rho(g_n^\perp))\| + \|\rho(f - g)\rho(fTf_n)(1 - \rho(g_n^\perp))\| = 0$, because $\rho(e^\perp) = \rho(f^\perp) = \rho(g^\perp) = 0$, as, for example, $\rho(e^\perp) = \tau \lim \rho(e_n^\perp)$. Indeed for all $\delta > 0$, let $k \in \mathbb{N}$ be s.t. $\tau(e_k^\perp) < \delta$, then for $n > k$ one has $\|\rho(e_n^\perp)(1 - \rho(e_k^\perp))\| = 0$. Finally $\|[\rho(eTe_n) - \rho(fTf_n)](1 - \rho(e_k^\perp))\| = 0$; therefore $\rho(T) \in \mathcal{M}$. ■

Observe that $\rho(\overline{eTe}) = \rho(T)$, because $\rho(\overline{eTe}) = \tau \lim \rho(eTe_n) = \rho(T)$.

THEOREM 3.13. *The map $\rho: \overline{\mathcal{R}} \rightarrow \overline{\mathcal{M}}$ is a morphism of almost everywhere bimodules. Therefore $\tau \circ \rho$ is a trace on $\overline{\mathcal{R}}$ as an almost everywhere bimodule on \mathcal{R} , extending τ on \mathcal{R} . Besides, if $S, T \in \overline{\mathcal{R}}$ and $\{e_n\}$ is a common SDD, then $\rho(SeT) = \rho(S)\rho(T)$.*

Proof. We divide it in steps.

(i) $T \in \overline{\mathcal{R}} \Rightarrow \rho(T^*) = \rho(T)^*$. Indeed

$$\begin{aligned} \rho(T^*) &= \tau \lim \rho(eT^*e_n) = \tau \lim \rho(eT^+e_n) = \tau \lim \rho(e_n T^+e) \\ &= \tau \lim \rho((eTe_n)^*) = \tau \lim \rho(eTe_n)^* = \rho(T)^*. \end{aligned}$$

(ii) $S, T \in \overline{\mathcal{R}} \Rightarrow \rho(S \oplus T) = \rho(S) \oplus \rho(T)$. Indeed using Theorem 3.3 in [25] one has

$$\begin{aligned} \rho(S \oplus T) &= \tau \lim \rho(g(S + T)^\natural g_n) = \tau \lim \rho(g(S + T) g_n) \\ &= \tau \lim \rho(gSg_n) \oplus \tau \lim \rho(gTg_n) = \rho(S) \oplus \rho(T). \end{aligned}$$

(iii) $a \in \mathcal{R}, T \in \overline{\mathcal{R}} \Rightarrow \rho(a \odot T) = \rho(a) \odot \rho(T)$, and $\rho(T \odot a) = \rho(T) \odot \rho(a)$. Assume first that $a \in \mathcal{R}$ is unitary. Let f_n be an SDD for T , and set $g_n := f_n \wedge af_n a^*$. Then, by [25, Theorem 3.7]

$$\begin{aligned} \rho(a \odot T) &= \tau \lim \rho(g(aT)^\natural g_n) = \tau \lim \rho(gaTg_n) = \tau \lim \rho(gafa^*aTf_n g_n) \\ &= \tau \lim (1 - \rho(g^\perp)) \rho(a) \rho(fTf_n)(1 - \rho(g_n^\perp)) = \rho(a) \odot \rho(T). \end{aligned}$$

The general case follows from (i) and (ii) in the proof of Theorem 3.10, and (ii) right above. The proof of $\rho(T \odot a) = \rho(T) \odot \rho(a)$ is analogous.

So it remains to prove the last statement of the theorem, which follows from

$$\begin{aligned} \rho(SeT) &= \tau \lim \rho(e_n SeTe_n) = \tau \lim \rho(e_n Se) \tau \lim \rho(eTe_n) \\ &= \rho(S) \rho(T). \quad \blacksquare \end{aligned}$$

Remark 3.14. We proved in Theorem 3.10 that $\overline{\mathcal{R}}/\sim$ is a $*$ -bimodule over \mathcal{R} , and it immediately follows from the definition of ρ contained in Proposition 3.12, that if $T \in \overline{\mathcal{R}}$, $T = 0$ τ -a.e., then $\rho(T) = 0$. Therefore we get a bimodule map from $\overline{\mathcal{R}}/\sim$ to $\overline{\mathcal{M}}$, which is not an isomorphism, in general. More precisely, given $T \in \overline{\mathcal{R}}$, then $\rho(T) = 0$ iff for any $\varepsilon > 0$, there is an SDD $\{e_n\}$ for T s.t. $\|eTe\| < \varepsilon$.

Finally we show how the previous construction can be applied in order to extend a semicontinuous semifinite trace on a concrete C^* -algebra to a suitable family of unbounded operators. Let $\mathcal{A} \subset \mathcal{B}(\mathcal{H})$ be a C^* -algebra with a semi-continuous semifinite trace τ , and let $\mathcal{A}^{\mathcal{R}}$ be the algebra of bounded Riemann measurable elements, then we call $\overline{\mathcal{A}^{\mathcal{R}}}$ the bimodule of unbounded Riemann measurable elements, and the extension of τ from $\mathcal{A}^{\mathcal{R}}$ to $\overline{\mathcal{A}^{\mathcal{R}}}$, provided by the previous Theorem, the (noncommutative) unbounded Riemann integral.

PROPOSITION 3.15. *Let (\mathcal{A}, τ) be a C^* -algebra with a semicontinuous semifinite trace τ , and $\mathcal{A}^{\mathcal{R}}$ and $\overline{\mathcal{A}^{\mathcal{R}}}$ be as above. Then τ extends to a trace on $\overline{\mathcal{A}^{\mathcal{R}}}$ as an almost everywhere bimodule on $\mathcal{A}^{\mathcal{R}}$, namely $\tau(uAu^*) = \tau(A)$ for any unitary operator $u \in \mathcal{A}^{\mathcal{R}}$, and any positive operator $A \in \overline{\mathcal{A}^{\mathcal{R}}}$. Moreover, unbounded Riemann functional calculi of bounded Riemann measurable elements are Riemann measurable, namely for any $x \in \mathcal{A}_{sa}^{\mathcal{R}}$ and $f \in \overline{\mathcal{R}_0}(\sigma(x) \setminus \{0\}, \mu_x)$, $f(x) \in \overline{\mathcal{A}^{\mathcal{R}}}$.*

Proof. The first statement follows by the previous results in this section. Let f be in $\overline{\mathcal{R}_0}(\sigma(x) \setminus \{0\}, \mu_x)$. By Proposition 6.5 in the Appendix, there exists an SDD given by the characteristic functions of Riemann measurable sets G_n s.t. $f|_{G_n}$ is Riemann measurable. As a consequence $\chi_{G_n}(x)$ give an SDD for $f(x)$, and the second statement follows. ■

Remark 3.16. (i) $\overline{\mathcal{A}^{\mathcal{R}}} \cap \mathcal{A}''$ is not an algebra, and it is larger than $\mathcal{A}^{\mathcal{R}}$, in general. However, given $x \in \overline{\mathcal{A}^{\mathcal{R}}} \cap \mathcal{A}''$, there is a projection $p \in \mathcal{A}^{\mathcal{R}}$ s.t. $\tau(p) = 0$, and $p^\perp x p^\perp \in \mathcal{A}^{\mathcal{R}}$. Namely x belongs to $\mathcal{A}^{\mathcal{R}}$ τ -a.e.

(ii) If $A \in \overline{\mathcal{A}^{\mathcal{R}}}$ has finite trace or is positive, its trace may be computed as $\lim_{e \in \mathcal{A}^{\mathcal{R}}} \tau(eAe)$, where the projections e satisfy $eAe \in \mathcal{A}^{\mathcal{R}}$.

4. SINGULAR TRACES ON C^* -ALGEBRAS

In this section we construct singular traces on a C^* -algebra with a semicontinuous semifinite trace. Let us first recall that, if \mathcal{M} is a von Neumann algebra with a normal semifinite faithful trace τ , $\overline{\mathcal{M}}$ the $*$ -algebra of

τ -measurable operators, and $T \in \overline{\mathcal{M}}$, its distribution function and non-increasing rearrangement, the basic building blocks for the construction of singular traces [10], are defined as (cf., e.g., [7, 10])

$$\lambda_T(t) := \tau(\chi_{(t, +\infty)}(|T|))$$

$$\mu_T(t) := \inf\{s \geq 0 : \lambda_T(s) \leq t\}.$$

Let now \mathcal{A} be a C^* -algebra with a semicontinuous semifinite trace τ acting on a Hilbert space \mathcal{H} . As follows from the previous Section, the GNS representation ρ of \mathcal{A} extends to a $*$ -bimodule map from the unbounded Riemann measurable operators in $\overline{\mathcal{A}^{\mathcal{R}}}$ into the measurable operators of $\overline{\mathcal{M} := \rho(\mathcal{A})''}$, so that we may define the distribution function (and therefore the associated non-increasing rearrangement) w.r.t. τ of an operator $T \in \overline{\mathcal{A}^{\mathcal{R}}}$ as $\lambda_T = \lambda_{\rho(T)}$, and we get $\mu_T = \mu_{\rho(T)}$. Let us observe that, if $T \in \overline{\mathcal{A}^{\mathcal{R}}}$ is a positive (unbounded) continuous functional calculus of an element in $\mathcal{A}^{\mathcal{R}}$, then $\chi_{(t, +\infty)}(T)$ belongs to $\mathcal{A}^{\mathcal{R}}$ a.e., therefore its distribution function may be defined without using the representation ρ as $\lambda_T(t) = \tau(\chi_{(t, +\infty)}(T))$. With these preliminaries out of the way, we may carry out the construction of singular traces (with respect to τ) as it has been done in [10].

DEFINITION 4.1. An operator $T \in \overline{\mathcal{A}^{\mathcal{R}}}$ is called *eccentric* at 0 if either

$$\int_0^1 \mu_T(t) dt < \infty \quad \text{and} \quad \liminf_{t \rightarrow 0} \frac{\int_0^{2t} \mu_T(s) ds}{\int_0^t \mu_T(s) ds} = 1$$

or

$$\int_0^1 \mu_T(t) dt = \infty \quad \text{and} \quad \limsup_{t \rightarrow 0} \frac{\int_{2t}^1 \mu_T(s) ds}{\int_t^1 \mu_T(s) ds} = 1.$$

It is called *eccentric* at ∞ if either

$$\int_1^\infty \mu_T(t) dt < \infty \quad \text{and} \quad \limsup_{t \rightarrow \infty} \frac{\int_{2t}^\infty \mu_T(s) ds}{\int_t^\infty \mu_T(s) ds} = 1$$

or

$$\int_1^\infty \mu_T(t) dt = \infty \quad \text{and} \quad \liminf_{t \rightarrow \infty} \frac{\int_1^{2t} \mu_T(s) ds}{\int_1^t \mu_T(s) ds} = 1.$$

The following proposition trivially holds

PROPOSITION 4.2. *Let (\mathcal{A}, τ) be a C^* -algebra with a semicontinuous semifinite trace, $T \in \overline{\mathcal{A}^{\mathcal{R}}}$, and let $X(T)$ denote the $*$ -bimodule over $\mathcal{A}^{\mathcal{R}}$ generated by T in $\overline{\mathcal{A}^{\mathcal{R}}}$, while $X(\rho(T))$ denotes the $*$ -bimodule over $\mathcal{M} := \rho(\mathcal{A})''$ generated by $\rho(T)$ in $\overline{\mathcal{M}}$. Then*

- (i) *T is eccentric if and only if $\rho(T)$ is*
- (ii) *$\rho(X(T)) \subset X(\rho(T))$.*

As in the case of von Neumann algebras, with any eccentric operator (at 0 or at ∞) in $\overline{\mathcal{A}^{\mathcal{R}}}$ we may associate a singular trace, where the word singular refers to the original trace τ . Indeed such singular traces vanish on τ -finite operators, and those associated to 0-eccentric operators even vanish on all bounded operators. Of course singular traces may be described as the pull-back of the singular traces on \mathcal{M} via the (extended) GNS representation. On the other hand, explicit formulas may be written in terms of the non-increasing rearrangement. Since Riemann integration is crucial in the extension of the trace to unbounded operators, we write these formulas only in case of 0-eccentric operators. Moreover this is the case occurring in Section 5.

THEOREM 4.3. *If $T \in \overline{\mathcal{A}^{\mathcal{R}}}$ is 0-eccentric and $\int_0^1 \mu_T(t) dt < \infty$, there exists a generalized limit Lim_ω in 0 such that the functional*

$$\tau_\omega(A) := \text{Lim}_\omega \left(\frac{\int_0^t \mu_A(s) ds}{\int_0^t \mu_T(s) ds} \right), \quad A \in X(T)_+$$

linearly extends to a singular trace on the a.e. $$ -bimodule $X(T)$ over $\mathcal{A}^{\mathcal{R}}$ generated by T , where $X(T)_+$ denotes those elements whose image under ρ is positive. If $\int_0^1 \mu_T(t) dt = \infty$, the previous formula should be replaced by*

$$\tau_\omega(A) := \text{Lim}_\omega \left(\frac{\int_t^1 \mu_A(s) ds}{\int_t^1 \mu_T(s) ds} \right), \quad A \in X(T)_+.$$

Such traces naturally extend to traces on $X(T) + \mathcal{A}^{\mathcal{R}}$.

We conclude this section mentioning that the proof of Lemma 2.5 in [10] contains a gap, even though the statement is correct. We thank B. De Pagter and F. Sukochev for having noticed this gap and having also furnished a correct proof. Since the Lemma is quite standard, stating that the unique positive dilation invariant functional on the cone of bounded, right continuous functions with compact support on $(0, +\infty)$ is the

Lebesgue measure (up to a positive constant), we do not include the proof here.

5. NOVIKOV–SHUBIN INVARIANTS AND SINGULAR TRACES

In this section we show how results developed in the previous sections can be applied to define and study Novikov–Shubin invariants on amenable (open) manifolds with bounded geometry. More precisely we assume that our manifold M

- is a complete Riemannian manifold
- has C^∞ -bounded geometry, i.e., it has positive injectivity radius, and curvature tensor bounded, with all its covariant derivatives
- is endowed with a regular exhaustion \mathcal{K} [20], that is with an increasing sequence $\{K_n\}$ of compact subsets of M , whose union is M , and such that, for any $r > 0$

$$\lim_{n \rightarrow \infty} \frac{\text{vol}(\text{Pen}^+(K_n, r))}{\text{vol}(\text{Pen}^-(K_n, r))} = 1,$$

where we set $\text{Pen}^+(K, r) := \{x \in M : \text{dist}(x, K) \leq r\}$, and $\text{Pen}^-(K, r) :=$ the closure of $M \setminus \text{Pen}^+(M \setminus K, r)$.

Let M be as above, F be a finite dimensional Hermitian vector bundle over M , and consider the C^* -algebra $\mathcal{A}(F)$ of almost local operators on $L^2(F)$, namely the norm closure of the $*$ -algebra of finite propagation operators, where $A \in \mathcal{B}(L^2(F))$ has finite propagation if there is a constant $u_A > 0$ s.t. for any compact subset K of M , any $\varphi \in L^2(F)$, $\text{supp } \varphi \subset K$, we have $\text{supp } A\varphi \subset \text{Pen}^+(K, u_A)$.

THEOREM 5.1 [12]. (i) $\mathcal{A}(F)$ contains all compact operators,

(ii) if $F = \Lambda^p T^*M$ is the bundle of p -forms on M , then $f(\Delta_p) \in \mathcal{A}_p := \mathcal{A}(\Lambda^p T^*M)$, for any $f \in C_0([0, \infty))$, where Δ_p is the p -Laplacian on M ,

(iii) there exists a semicontinuous semifinite (non-densely defined) trace $\text{Tr}_{\mathcal{K}}$ on $\mathcal{A}(F)$, which vanishes on compact operators and, on the set of uniform operators of order $-\infty$ [20], is finite, and assumes the form

$$\text{Tr}_{\mathcal{K}}(A) = \text{Lim}_{\omega} \frac{\int_{K_n} \text{tr}(a(x, x)) \, d \, \text{vol}(x)}{\text{vol}(K_n)},$$

where $a(x, y)$ is the kernel of A .

5.1. Novikov–Shubin Numbers for Open Manifolds and Their Invariance

In this subsection we define the Novikov–Shubin numbers for the mentioned class of manifolds and prove their invariance under quasi-isometries.

Applying the results of Section 2 to \mathcal{A}_p , we obtain the C^* -algebra $\mathcal{A}_p^{\mathcal{R}}$ with a lower-semicontinuous semifinite trace, still denoted $Tr_{\mathcal{K}}$. Then $\chi_{[0, t)}(\Delta_p)$ and $\chi_{[\varepsilon, t)}(\Delta_p)$ belong to $\mathcal{A}_p^{\mathcal{R}}$ for almost all $t > \varepsilon > 0$, by Proposition 2.3. Denote by $N_p(t) := Tr_{\mathcal{K}}(\chi_{[0, t)}(\Delta_p))$, $\mathfrak{P}_p(t) := Tr_{\mathcal{K}}(e^{-t\Delta_p})$.

LEMMA 5.2. $\mathfrak{P}_p(t) = \int_0^\infty e^{-t\lambda} dN_p(\lambda)$ so that $\lim_{t \rightarrow 0} N_p(t) = \lim_{t \rightarrow \infty} \mathfrak{P}_p(t)$.

Proof. If $\Delta_p = \int_0^\infty \lambda de_p(\lambda)$ denotes the spectral decomposition, then $e^{-t\Delta_p} = \int_0^\infty e^{-t\lambda} de_p(\lambda)$. Since the latter is defined as the norm limit of the Riemann–Stieltjes sums, $\pi_p(e^{-t\Delta_p}) = \int_0^\infty e^{-t\lambda} d\pi_p(e_p(\lambda))$, where π_p denotes the GNS representation of \mathcal{A}_p w.r.t. the trace $Tr_{\mathcal{K}}$. The result then follows by the normality of the trace in the GNS representation. ■

DEFINITION 5.3. We define $b_p \equiv b_p(M, \mathcal{K}) := \lim_{t \rightarrow 0} N_p(t) = \lim_{t \rightarrow \infty} \mathfrak{P}_p(t)$ to be the p th L^2 -Betti number of the open manifold M endowed with the exhaustion \mathcal{K} . Let us now set $N_p^0(t) := N_p(t) - b_p \equiv \lim_{\varepsilon \rightarrow 0} Tr_{\mathcal{K}}(\chi_{[\varepsilon, t)}(\Delta_p))$, and $\mathfrak{P}_p^0(t) := \mathfrak{P}_p(t) - b_p = \int_0^\infty e^{-t\lambda} dN_p^0(\lambda)$. The Novikov–Shubin numbers of (M, \mathcal{K}) are then defined as

$$\alpha_p \equiv \alpha_p(M, \mathcal{K}) := 2 \limsup_{t \rightarrow 0} \frac{\log N_p^0(t)}{\log t},$$

$$\underline{\alpha}_p \equiv \underline{\alpha}_p(M, \mathcal{K}) := 2 \liminf_{t \rightarrow 0} \frac{\log N_p^0(t)}{\log t},$$

$$\alpha'_p \equiv \alpha'_p(M, \mathcal{K}) := 2 \limsup_{t \rightarrow \infty} \frac{\log \mathfrak{P}_p^0(t)}{\log 1/t},$$

$$\underline{\alpha}'_p \equiv \underline{\alpha}'_p(M, \mathcal{K}) := 2 \liminf_{t \rightarrow \infty} \frac{\log \mathfrak{P}_p^0(t)}{\log 1/t}.$$

It follows from [9, Appendix] that $\underline{\alpha}_p = \underline{\alpha}'_p \leq \alpha'_p \leq \alpha_p$, and $\alpha'_p = \alpha_p$ if $\mathfrak{P}_p^0(t) = O(t^{-\delta})$, for $t \rightarrow \infty$, or equivalently $N_p^0(t) = O(t^\delta)$, for $t \rightarrow 0$. Observe that L^2 -Betti numbers and Novikov–Shubin numbers depend on the limit procedure ω and the exhaustion \mathcal{K} .

Remark 5.4. (a) The L^2 -Betti numbers for amenable manifolds of bounded geometry have been defined by Roe [20], and it is easy to show that the two definitions agree (see [12]). Moreover Roe proved [22] that they are invariant under quasi-isometries (see below).

(b) If M is a covering of a compact manifold X , L^2 -Betti numbers were introduced by Atiyah [1] whereas Novikov–Shubin numbers were introduced in [17]. They were proved to be Γ -homotopy invariants, where $\Gamma := \pi_1(X)$ is the fundamental group of X , by Dodziuk [6] and Gromov and Shubin [9], respectively.

(c) In the case of coverings, the trace Tr_Γ is normal on the von Neumann algebra of Γ -invariant operators, hence $\lim_{t \rightarrow 0} Tr(e_{[0,t)}(\Delta_p)) = Tr(e_{\{0\}}(\Delta_p))$. In the case of open manifolds there is no natural von Neumann algebra containing the bounded functional calculi of Δ_p on which the trace $Tr_{\mathcal{K}}$ is normal, hence the previous equality does not necessarily hold. Such phenomenon was already noticed by Roe [21]. It has been considered by Farber in [8] in a more general context, and the difference $\lim_{t \rightarrow 0} Tr(e_{[0,t)}(\Delta_p)) - Tr(e_{\{0\}}(\Delta_p))$ has been called the *torsion dimension*. We shall denote by $\text{tordim}(M, \Delta_p)$ such difference.

(d) Let us observe that the above definitions for L^2 -Betti numbers and Novikov–Shubin numbers coincide with the classical ones in the case of amenable coverings, if one chooses the exhaustion given by the Følner condition. An explicit argument is given in [11].

We prove now that Novikov–Shubin numbers are invariant under quasi-isometries, where a map $\varphi: M \rightarrow \tilde{M}$ between open manifolds of C^∞ -bounded geometry is a quasi-isometry [22] if φ is a diffeomorphism s.t.

(i) there are $C_1, C_2 > 0$ s.t. $C_1 \|v\| \leq \|\varphi_* v\| \leq C_2 \|v\|$, $v \in TM$

(ii) $\nabla - \varphi^* \tilde{\nabla}$ is bounded with all its covariant derivatives, where $\nabla, \tilde{\nabla}$ are Levi-Civita connections of M and \tilde{M} .

THEOREM 5.5. *Let (M, \mathcal{K}) be an open manifold of bounded geometry with a regular exhaustion, and let $\varphi: M \rightarrow \tilde{M}$ be a quasi-isometry. Then $\varphi(\mathcal{K})$ is a regular exhaustion for \tilde{M} , $\alpha_p(M, \mathcal{K}) = \alpha_p(\tilde{M}, \varphi(\mathcal{K}))$ and the same holds for α'_p and α'_p .*

Proof. Let us denote by $\Phi \in \mathcal{B}(L^2(\Lambda^* T^* M), L^2(\Lambda^* T^* \tilde{M}))$ the extension of $(\varphi^{-1})^*$. Then $Tr_{\varphi(\mathcal{K})} = Tr_{\mathcal{K}}(\Phi^{-1} \cdot \Phi)$. Also, setting $e_{\varepsilon, t} := \chi_{[\varepsilon, t)}(\Delta_p)$, $q_{\eta, s} := \Phi^{-1} \chi_{[\eta, s)}(\tilde{\Delta}_p) \Phi$, we have

$$\begin{aligned} 0 &\leq Tr_{\mathcal{K}}(e_{\varepsilon, t} - e_{\varepsilon, t} q_{\eta, s} e_{\varepsilon, t}) = Tr_{\mathcal{K}}(e_{\varepsilon, t} (1 - q_{\eta, s}) e_{\varepsilon, t}) \\ &= Tr_{\mathcal{K}}(e_{\varepsilon, t} q_{0, \eta} e_{\varepsilon, t}) + Tr_{\mathcal{K}}(e_{\varepsilon, t} q_{s, \infty} e_{\varepsilon, t}) \\ &= Tr_{\mathcal{K}}(q_{0, \eta} e_{\varepsilon, t} q_{0, \eta}) + Tr_{\mathcal{K}}(e_{\varepsilon, t} e_{0, t} q_{s, \infty} e_{0, t}) \\ &\leq Tr_{\mathcal{K}}(q_{0, \eta} e_{\varepsilon, \infty} q_{0, \eta}) + Tr_{\mathcal{K}}(e_{\varepsilon, t} e_{0, t} q_{s, \infty} e_{0, t}) \end{aligned}$$

$$\begin{aligned}
&\leq \operatorname{Tr}_{\mathcal{K}}(q_{0,\eta}) \|q_{0,\eta} e_{\varepsilon,\infty} q_{0,\eta}\| + \operatorname{Tr}_{\mathcal{K}}(e_{\varepsilon,t}) \|e_{0,t} q_{s,\infty} e_{0,t}\| \\
&\leq \operatorname{Tr}_{\mathcal{K}}(q_{0,\eta}) C \sqrt{\frac{\eta}{\varepsilon}} + \operatorname{Tr}_{\mathcal{K}}(e_{\varepsilon,t}) C \sqrt{\frac{t}{s}},
\end{aligned}$$

where the last inequality follows from [22]. Then

$$\begin{aligned}
\operatorname{Tr}_{\mathcal{K}}(q_{\eta,s}) &= \operatorname{Tr}_{\mathcal{K}}(e_{\varepsilon,t}) + \operatorname{Tr}_{\mathcal{K}}(q_{\eta,s} - e_{\varepsilon,t} q_{\eta,s} e_{\varepsilon,t}) - \operatorname{Tr}_{\mathcal{K}}(e_{\varepsilon,t} - e_{\varepsilon,t} q_{\eta,s} e_{\varepsilon,t}) \\
&\geq \operatorname{Tr}_{\mathcal{K}}(e_{\varepsilon,t}) - \operatorname{Tr}_{\mathcal{K}}(q_{0,\eta}) C \sqrt{\frac{\eta}{\varepsilon}} - \operatorname{Tr}_{\mathcal{K}}(e_{\varepsilon,t}) C \sqrt{\frac{t}{s}}.
\end{aligned}$$

Now let $a > 1$ and compute

$$\begin{aligned}
\tilde{N}^0(s) &= \lim_{\varepsilon \rightarrow 0} \operatorname{Tr}_{\mathcal{K}}(q_{\varepsilon^a,s}) \\
&\geq \lim_{\varepsilon \rightarrow 0} \left[\operatorname{Tr}_{\mathcal{K}}(e_{\varepsilon,t}) - \operatorname{Tr}_{\mathcal{K}}(q_{0,\varepsilon^a}) C \varepsilon^{(a-1)/2} - \operatorname{Tr}_{\mathcal{K}}(e_{\varepsilon,t}) C \sqrt{\frac{t}{s}} \right] \\
&= N^0(t) \left[1 - C \sqrt{\frac{t}{s}} \right].
\end{aligned}$$

Therefore with $\lambda := 4C^2$ we get $\tilde{N}^0(\lambda t) \geq \frac{1}{2}N^0(t)$, and exchanging the roles of M and \tilde{M} , we obtain $\frac{1}{2}N^0(\lambda^{-1}t) \leq \tilde{N}^0(t) \leq 2N^0(\lambda t)$. This means that N^0 and \tilde{N}^0 are dilatation-equivalent (see [9]) so that the thesis follows from [9]. ■

Remark 5.6. We have chosen Lott's normalization [15] for the Novikov–Shubin numbers $\alpha_p(M)$, instead of the original one in [17]. In contrast with Lott's choice, we used the lim sup in Definition 5.3. This is motivated by our interpretation of $\alpha_p(M)$ as a dimension, as a noncommutative measure corresponds to α_p via a singular trace, according to Theorem 5.10.

In [12] an asymptotic dimension is defined for any (noncompact) metric space. For a suitable class of open manifolds it is shown to coincide with $\alpha_0(M)$. Therefore in this case $\alpha_0(M)$ is independent of the exhaustion and the limit procedure.

5.2. Novikov–Shubin Numbers as Asymptotic Spectral Dimensions

In this subsection we show that Novikov–Shubin numbers can be interpreted as noncommutative asymptotic dimensions. More precisely, we prove that $\alpha_p(M, \mathcal{K})$ can be expressed by a formula which is a large scale analogue of Weyl's asymptotic formula for the dimension of a manifold

$$\alpha_p(M, \mathcal{K}) = \left(\liminf_{t \rightarrow 0} \frac{\log \mu_p(t)}{\log 1/t} \right)^{-1},$$

where μ_p refers to the operator $\Delta_p^{-1/2}$. When α_p is finite non-zero, $\Delta_p^{-\alpha_p/2}$ gives rise to a singular trace, namely there exists a type II_1 singular trace which is finite nonzero on $\Delta_p^{-\alpha_p/2}$.

This result, which extends the analogous result for coverings in [11], makes essential use of the unbounded Riemann integration and the theory of singular traces for C^* -algebras developed in Sections 3 and 4. However, since the trace we use is not normal with respect to the given representation of \mathcal{A}_p on the space of L^2 -differential forms, some assumptions, like the vanishing of the torsion dimension introduced in Remark 5.4(c), are needed.

In the following, when the Laplacian Δ_p has a non-trivial kernel, we denote by $\Delta_p^{-\alpha}$, $\alpha > 0$, the (unbounded) functional calculus of Δ_p w.r.t. the function φ_α given by $\varphi_\alpha(0) = 0$ and $\varphi_\alpha(t) = t^{-\alpha}$ when $t > 0$.

PROPOSITION 5.7. *Let M be an amenable open manifold. If*

(a) *the projection E_p onto the kernel of Δ_p is Riemann measurable, and the torsion dimension vanishes, namely $\text{Tr}_{\mathcal{K}}(E_p)$ is equal to b_p , then $\Delta_p^{-\alpha} \in \overline{\mathcal{A}_p^{\mathcal{R}}}$ for any $\alpha > 0$.*

The vanishing of the Betti number b_p implies (a). It is equivalent to (a) if $\text{Ker}(\Delta_p)$ is finite-dimensional.

Proof. By hypothesis $E_p = \chi_{\{1\}}(e^{-\Delta_p}) \in \mathcal{A}_p^{\mathcal{R}}$, hence $T_p := \chi_{[0,1)}(e^{-\Delta_p}) e^{-\Delta_p} = e^{-\Delta_p} - E_p \in \mathcal{A}_p^{\mathcal{R}}$. Then the spectral measure ν_p associated with T_p as in Eq. (2.2), is a finite measure on $[0, 1]$ (see Theorem 5.1) and $\nu_p(\{1\}) = 0$. Therefore, the function

$$f_\alpha(t) := \begin{cases} (-\log t)^{-\alpha}, & t \in (0, 1) \\ 0, & t = 0, 1, \end{cases}$$

belongs to $\overline{\mathcal{R}_0}((0, 1], \nu_p)$ (see Proposition 6.5), and $\Delta_p^{-\alpha} = f_\alpha(T_p) = f_\alpha(e^{-\Delta_p}) \in \overline{\mathcal{A}_p^{\mathcal{R}}}$ by Proposition 3.15. This proves the first statement.

If $\lim_{t \rightarrow \infty} \text{Tr}_{\mathcal{K}}(e^{-t\Delta_p}) = 0$, from $0 \leq \chi_{\{1\}}(e^{-\Delta_p}) \leq e^{-t\Delta_p}$, we have that $\chi_{\{1\}}(e^{-\Delta_p})$ is a separating element for an \mathcal{R} -cut in \mathcal{A} . The last statement follows from the vanishing of $\text{Tr}_{\mathcal{K}}$ on compact operators (see Theorem 5.1). ■

Conditions implying the vanishing of L^2 -Betti numbers are given in [16].

If hypothesis (a) of the previous lemma is satisfied, we may define the distribution function λ_p and the eigenvalue function μ_p for the operator $\Delta_p^{-1/2}$, hence the local spectral dimension as the inverse of $\lim_{t \rightarrow \infty} (\log \mu_p(t) / \log 1/t)$, which may be shown to coincide with the dimension of the manifold for any p . Moreover

DEFINITION 5.8. The asymptotic spectral dimension of the triple $(M, \mathcal{K}, \Delta_p)$ is

$$\left(\liminf_{t \rightarrow 0} \frac{\log \mu_p(t)}{\log 1/t} \right)^{-1}.$$

Remark 5.9. (i) The extension of the GNS representation π to $\mathcal{A}_p^{\mathcal{R}}$ does not necessarily commute with the Borel functional calculus. In particular $\chi_{\{1\}}(\pi(e^{-\Delta_p}))$ is not necessarily equal to $\pi(\chi_{\{1\}}(e^{-\Delta_p}))$.

(ii) Condition (a) of the previous proposition is equivalent to

(a') $\chi_{\{1\}}(\pi(e^{-\Delta_p}))$ is Riemann integrable in the GNS representation π .

The proof goes as follows.

(a) \Rightarrow (a'). Since the projection $E_p \equiv \chi_{\{0\}}(\Delta_p)$ is Riemann integrable and less than $e^{-t\Delta_p}$ for any t , its image in the GNS representation is Riemann integrable and less than $\pi(e^{-t\Delta_p})$ for any t . This implies that $\pi(E_p) \leq \chi_{\{1\}}(\pi(e^{-\Delta_p})) \leq \pi(e^{-t\Delta_p})$ is an \mathcal{R} -cut, hence the thesis.

(a') \Rightarrow (a). By normality of the trace in the GNS representation, $Tr_{\mathcal{K}}(e^{-t\Delta_p})$ converges to $Tr_{\mathcal{K}}(\chi_{\{1\}}(\pi(e^{-\Delta_p})))$ hence, by hypothesis, for any $\varepsilon > 0$ we may find $a_\varepsilon \in \mathcal{A}$ and $t_\varepsilon > 0$ s.t. $a_\varepsilon \leq \chi_{\{1\}}(\pi(e^{-\Delta_p}))$ and $Tr_{\mathcal{K}}(e^{-t_\varepsilon \Delta_p} - a_\varepsilon) < \varepsilon$. This implies $a_\varepsilon \leq e^{-t_\varepsilon \Delta_p}$ for any t , hence $a_\varepsilon \leq \chi_{\{0\}}(\Delta_p)$, which means that $(\{a_\varepsilon\}, \{e^{-t_\varepsilon \Delta_p}\})$ is an \mathcal{R} -cut for $\chi_{\{0\}}(\Delta_p)$, namely this projection is Riemann integrable and $Tr_{\mathcal{K}}(e^{-t_\varepsilon \Delta_p} - \chi_{\{0\}}(\Delta_p)) \leq \varepsilon$, i.e., the thesis.

THEOREM 5.10. Let (M, \mathcal{K}) be an open manifold equipped with a regular exhaustion such that the projection on the kernel of Δ_p is Riemann integrable and $\text{tordim}(M, \Delta_p) = 0$. Then

(i) the asymptotic spectral dimension of $(M, \mathcal{K}, \Delta_p)$ coincides with the Novikov-Shubin number $\alpha_p(M, \mathcal{K})$,

(ii) if α_p is finite nonzero, then $\Delta_p^{-\alpha_p/2}$ is 0-eccentric, therefore gives rise to a non trivial singular trace on the unbounded Riemann measurable operators of \mathcal{A}_p .

Proof. (i) By hypothesis, $e_{(0,t)}(\Delta_p)$ is Riemann integrable $Tr_{\mathcal{K}}$ -a.e., hence $N_p^0(t) = Tr_{\mathcal{K}}(e_{(0,t)}(\Delta_p)) = Tr_{\mathcal{K}}(e_{(t^{-1}, \infty)}(\Delta_p^{-1})) = Tr_{\mathcal{K}}(e_{(t^{-1/2}, \infty)}(\Delta_p^{-1/2})) = \lambda_p(t^{-1/2})$. Then

$$\begin{aligned} \alpha_p &= 2 \limsup_{s \rightarrow 0} \frac{\log N_p^0(s)}{\log s} = 2 \limsup_{s \rightarrow 0} \frac{\log \lambda_p(s^{-1/2})}{\log s} \\ &= \limsup_{t \rightarrow \infty} \frac{\log \lambda_p(t)}{\log(1/t)}. \end{aligned} \quad (5.1)$$

The statement follows from

$$\liminf_{t \rightarrow 0} \frac{\log \mu(t)}{\log(1/t)} = \left(\limsup_{s \rightarrow \infty} \frac{\log \lambda(s)}{\log(1/s)} \right)^{-1}$$

which is proved in [11].

(ii) When $0 < \alpha_p < \infty$,

$$\liminf_{t \rightarrow 0} \frac{\log \mu_{\Delta_p^{-\alpha_p/2}}(t)}{\log(1/t)} = 1,$$

and this implies the eccentricity condition, as shown in [11]. Hence the thesis follows by Theorem 4.3. ■

6. APPENDIX

Here we present some more or less known results on the Riemann measurable functions on a locally compact Hausdorff space X which are needed in the previous sections.

Let us introduce $\ell_0^\infty(X) := \{f: X \rightarrow \mathbb{C}: f \text{ is bounded and } \lim_{x \rightarrow \infty} f(x) = 0\}$, and, for μ an outer regular, complete, positive Borel measure, $\mathcal{R}_0(X, \mu) := \{f \in \ell_0^\infty(X): f \text{ is continuous but for a set of zero } \mu\text{-measure}\}$, the set of Riemann μ -measurable functions. Let us observe that any semicontinuous semifinite trace on $C_0(X)$ gives rise to such a measure μ .

A different description of $\mathcal{R}_0(X, \mu)$ is contained in the following proposition, whose proof we leave to the reader.

PROPOSITION 6.1. *Let $f: X \rightarrow \mathbb{R}$, then the following are equivalent*

(i) $f \in \mathcal{R}_0(X, \mu)$

(ii) for any $\varepsilon > 0$ there are $f_\varepsilon^\pm \in C_0(X)$ s.t. $f_\varepsilon^- \leq f \leq f_\varepsilon^+$ and $\int (f_\varepsilon^+ - f_\varepsilon^-) d\mu < \varepsilon$

(iii) there are $h \in C_0(X)$, an open subset V of finite μ -measure, and, for any $\varepsilon > 0$, $h_\varepsilon^\pm \in C_0(V)$ s.t. $h_\varepsilon^- \leq f - h \leq h_\varepsilon^+$ and $\int (h_\varepsilon^+ - h_\varepsilon^-) d\mu < \varepsilon$.

We now prove some lemmas used in Section 2.

A measurable set $\Omega \subset X$ is said Riemann μ -measurable if its characteristic function is Riemann μ -measurable, which is equivalent to saying $\mu(\partial\Omega) = 0$.

SUBLEMMA 6.2. *Let f be a positive Riemann μ -measurable function on X such that $\int f < \infty$ and set $O_y = \{x \in X : f(x) > y\}$, $C_y = \{x \in X : f(x) \geq y\}$, $y \geq 0$. Then, for any $0 < \alpha < \beta$ there are uncountably many $y \in (\alpha, \beta)$ for which O_y and C_y are Riemann μ -measurable.*

We omit the proof since it follows by standard arguments.

LEMMA 6.3. *Let f be a positive Riemann measurable function such that $\int f d\mu < \infty$. Then, for any $\delta > 0$, we may find a sequence of Riemann measurable characteristic functions χ_n and a sequence of positive numbers α_n such that $\sum_n \alpha_n = \|f\|$ and $f = \sum_n \alpha_n \chi_n$.*

Proof. First we construct by induction the sequences $\chi_n := \chi_{\Omega_n}$ and $\alpha_n > 0$, $n \geq 1$, such that, $\forall n \geq 0$,

- $f \geq \sum_1^n \alpha_k \chi_k$,
- $\beta_n := \|f - \sum_1^n \alpha_k \chi_k\| > 0$,
- $\beta_n/4 \leq \alpha_{n+1} \leq \beta_n/2$,
- $\Omega_{n+1} := \{x \in X : f(x) - \sum_1^n \alpha_k \chi_k(x) \geq \alpha_{n+1}\}$ is Riemann measurable.

Indeed, given χ_k, α_k for $k \leq n$, as prescribed, the existence of α_{n+1} satisfying the last two properties follows by Sublemma 6.2, while the first two inequalities (hence $\alpha_n > 0$) follow by the definition of α_{n+1} and Ω_{n+1} .

We now observe that, since $\sup_X (f - \sum_1^n \alpha_k \chi_k) = \sup_{\Omega_{n+1}} (f - \sum_1^n \alpha_k \chi_k)$, we have

$$\begin{aligned} \sup_{\Omega_{n+1}} \left(f - \sum_{k=1}^{n+1} \alpha_k \chi_k \right) &= \sup_{\Omega_{n+1}} \left(f - \sum_{k=1}^n \alpha_k \chi_k - \alpha_{n+1} \right) \\ &= \beta_n - \alpha_{n+1} \geq \alpha_{n+1} \\ \sup_{\Omega_{n+1}^c} \left(f - \sum_{k=1}^{n+1} \alpha_k \chi_k \right) &= \sup_{\Omega_{n+1}^c} \left(f - \sum_{k=1}^n \alpha_k \chi_k \right) \leq \alpha_{n+1}, \end{aligned}$$

hence

$$\begin{aligned} \beta_{n+1} &= \max \left(\sup_{\Omega_{n+1}} \left(f - \sum_{k=1}^{n+1} \alpha_k \chi_k \right), \sup_{\Omega_{n+1}^c} \left(f - \sum_{k=1}^{n+1} \alpha_k \chi_k \right) \right) \\ &= \beta_n - \alpha_{n+1} \leq \frac{3}{4} \beta_n. \end{aligned}$$

This shows at once that $\beta_n \leq (3/4)^n \|f\| \rightarrow 0$, namely $\sum \alpha_n \chi_n$ converges to f uniformly and $\sum_0^\infty \alpha_{n+1} = \sum_0^\infty (\beta_n - \beta_{n+1}) = \|f\|$, which concludes the proof. \blacksquare

LEMMA 6.4. *Let $\Omega \subset\subset X$ with $\mu(\partial\Omega)=0$ and $\mu(\Omega)<\infty$. Then for any $\varepsilon>0$ $\exists f_\varepsilon^\pm \in C_0(X)$ such that $0 \leq f_\varepsilon^- \leq \chi_\Omega \leq f_\varepsilon^+ \leq 1$, $\int (f_\varepsilon^+ - f_\varepsilon^-) d\mu \leq \varepsilon$ and $\mu(\text{supp}(f_\varepsilon^+) \setminus \Omega) \leq \varepsilon$.*

Proof. Since χ_Ω is Riemann measurable, by Proposition 6.1, we may find f_ε^\pm satisfying all the properties above, except possibly the last one. Then, choosing a continuous increasing function ψ_δ on $[0, 1]$ s.t. $\psi_\delta(t)=0$ when $t \in [0, 1-\delta]$ and $\psi_\delta(1)=1$, we may replace f_ε^+ with $\psi_\delta \circ f_\varepsilon^+$. We have

$$\begin{aligned} \mu(\text{supp}(\psi_\delta \circ f_\varepsilon^+)) &\leq \mu(\{x \in X : f_\varepsilon^+(x) > 1-\delta\}) \\ &\leq \frac{1}{1-\delta} \int f_\varepsilon^+ d\mu \leq \frac{1}{1-\delta} (\varepsilon + \mu(\Omega)) \end{aligned}$$

from which the thesis follows. \blacksquare

We conclude this Appendix giving a characterization, in the commutative case, of the unbounded Riemann μ -measurable functions.

PROPOSITION 6.5. *Setting $\bar{\mathcal{R}}_0(X, \mu) := \{f: X \rightarrow \mathbb{C} : f \text{ is } \mu\text{-a.e. defined and continuous, and there is a compact, Riemann } \mu\text{-measurable subset } K \text{ of finite } \mu\text{-measure s.t. } f|_{K^c} \in \ell_0^\infty(X)\}$, we have $\bar{\mathcal{R}}_0(X, \mu) = \overline{(C_0(X))^\mathcal{R}}$, in the universal algebraic representation.*

Proof. Let $f \in \overline{(C_0(X))^\mathcal{R}}$. Then there is an increasing sequence of Riemann μ -measurable subsets G_n s.t. $G_n^c \subset\subset X$, $\mu(G_n^c) \searrow 0$ and $f|_{G_n} \in \mathcal{R}_0(X, \mu)$. Therefore $\lim_{x \rightarrow \infty} f(x) = 0$, and, for any $n \in \mathbb{N}$ there is $E_n \subset G_n$ s.t. $\mu(E_n) = 0$ and $f|_{E_n^c \cap G_n}$ is continuous. Setting $G := \bigcup_n G_n$, $E := \bigcup_n E_n$, we get $\mu(E) = \mu(G^c) = 0$ and $f|_{E^c \cap G}$ is (defined and) continuous. Finally choose an $n \in \mathbb{N}$ and set $K := \overline{G_n^c}$. Then $f \in \bar{\mathcal{R}}_0(X, \mu)$.

Let now $f \in \bar{\mathcal{R}}_0(X, \mu)$, and let $E \subset X$ be s.t. $f|_{E^c}$ is defined and continuous, and $\mu(E) = 0$. Then, because of outer regularity of μ , there are open sets $\Omega_\varepsilon \supset E$, with $\mu(\Omega_\varepsilon) < \varepsilon$. Let us fix $\varepsilon > 0$ and set, for any $\lambda > 0$, $V_{\varepsilon, \lambda} := \{x \in K \cap \Omega_\varepsilon^c : |f(x)| \leq \lambda\}$. Then, with $M := \sup_{K^c} |f|$, from Sublemma 6.2 we conclude the existence of uncountably many $\lambda > M$ s.t. $V_{\varepsilon, \lambda}$ is Riemann μ -measurable. For any $\varepsilon > 0$ choose one such $\lambda \equiv \lambda_\varepsilon$ so large that, with $A_\varepsilon := V_{\varepsilon, \lambda_\varepsilon}^o$, we get $\mu(K \cap A_\varepsilon^c) = \mu(K) - \mu(V_{\varepsilon, \lambda_\varepsilon}) < 2\varepsilon$. Finally set $G_n := \bigcup_{k=1}^n A_{1/k} \cup K^c$, which is an increasing family of Riemann μ -measurable subsets, s.t. $\mu(G_n^c) \leq \mu(K \cap A_{1/n}^c) \leq \frac{2}{n}$, and $\sup_{G_n} |f| = \max\{M, \max_{k \leq n} \sup_{A_{1/k}} |f|\} \leq \max_{k \leq n} \lambda_{1/k}$. Therefore $\{G_n\}$ is an SDD for f , and $f \in \overline{(C_0(X))^\mathcal{R}}$. \blacksquare

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